EUCLID'S

ELEMENTS OF GEOMETRY

BOOKS I—VI.

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EUCLID'S

ELEMENTS OF GEOMETRY

EDITED FOR THE SYNDICS OF THE PRESS

BY

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BOOKS I—VI.

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NOTE.

The Special Board for Mathematics in the University of Cambridge in a Report on Geometrical Teaching dated May 10, 1887, state as follows:

'The majority of the Board are of opinion that the rigid adherence to Euclid's texts is prejudicial to the interests of education, and that greater freedom in the method of teaching Geometry is desirable. As it appears that this greater freedom cannot be attained while a knowledge of Euclid's text is insisted upon in the examinations of the University, they consider that such alterations should be made in the regulations of the examinations as to admit other proofs besides those of Euclid, while following however his general sequence of propositions, so that no proof of any proposition occurring in Euclid should be accepted in which a subsequent proposition in Euclid's order is assumed.'

On March 8, 1888, Amended Regulations for the Previous Examination, which contained the following provision, were approved by the Senate:

'Euclid's definitions will be required, and no axioms or postulates except Euclid's may be assumed. The actual proofs of propositions as given in Euclid will not be required, but no proof of any proposition occurring in Euclid will be admitted in which use is made of any proposition which in Euclid's order occurs subsequently.'

And in the Regulations for the Local Examinations conducted by the University of Cambridge it is provided that:

'Proofs other than Euclid's will be admitted, but Euclid's Axioms will be required, and no proof of any proposition will be accepted which assumes anything not proved in preceding propositions in Euclid.'

PREFACE TO BOOKS I. AND II.

IT was with extreme diffidence that I accepted an invitation from the Syndics of the Cambridge University Press to undertake for them a new edition of the Elements of Euclid. Though I was deeply sensible of the honour, which the invitation conferred, I could not but recognise the great responsibility, which the acceptance of it would entail.

The invitation of the Syndics was in itself, to my mind, a sign of a widely felt conviction that the editions in common use were capable of improvement. Now improvement necessitates change, and every change made in a work, which has been a text book for centuries, must run the gauntlet of severe criticism, for while some will view every alteration with aversion, others will consider that every change demands an apology for the absence of more and greater changes.

I will here give a short account of the chief points, in which this edition differs from the best known editions of the Elements of Euclid at present in use in England.

While the texts of the editions of Potts and Todhunter are confessedly little more than reprints of Simson's English version of the Elements published in 1756, the text of the present edition does not profess to be a *translation* from the Greek. I began by retranslating the First Book: but there proved to be so many points, in which I thought it desirable to depart from the original, that it seemed best to give up all idea of simple translation and to retain merely the substance of the work, following closely Euclid's sequence of Propositions in Books I. and II. at all events.

Some of the definitions of Euclid, for instance *trapezium*, *rhomboid*, *gnomon* are omitted altogether as unnecessary. The word *trapezium* is defined in the Greek to mean "*any four sided figure other than those already defined*," but in many modern works it is defined to be "*a quadrilateral, which has one pair of parallel sides.*" The first of these definitions is obsolete, the second is not universally accepted. On the other hand definitions are added of several words in general use, such as *perimeter*, *parallelogram*, *diagonal*, which do not occur in Euclid's list.

The chief alteration in the definitions is in that of the word *figure*, which is in the Greek text defined to be "*that which is enclosed by one or more boundaries.*" I have preferred to define a *figure* as "*a combination of points, lines and surfaces.*" That Euclid's definition leads to difficulty is seen from the fact that, though Euclid defines a circle as "*a figure contained by one line*...", he demands in his postulate that "*a circle may be described*...". Now it is the circumference of a circle which is described and not the surface. Again, when two circles intersect, it is the circumferences which intersect and not the surfaces.

I have rejected the ordinarily received definition of a square as "*a quadrilateral, whose sides are equal, and whose angles are right angles.*" There is no doubt that, when we define any geometrical figure, we postulate the possibility of the figure; but it is useless to embrace in the definition more properties than are requisite to determine the figure.

The word axiom is used in many modern works as applicable both to simple *geometrical* propositions, such as "*two straight lines cannot enclose a space*," and to propositions, other than geometrical, accepted without demonstration and true *universally*, such as "*the whole of a thing is greater than a part.*" These two classes of propositions are often distinguished by the terms "geometrical axioms" and "general axioms." I prefer to use the word *axiom* as applicable to the latter class only, that is, to simple propositions, true of magnitudes of all kinds (for instance "*things which are equal to the same thing are equal to one another*"), and to use the term *postulate* for a simple *geometrical* proposition, whose truth we assume.

When a child is told that A weighs exactly as much as B, and B weighs exactly as much as C, he without hesitation arrives at the conclusion that A weighs exactly as much as C. His conviction of the validity of his conclusion would not be strengthened, and possibly his confidence in his conclusion might be impaired, by his being directed to appeal to the authority of the general proposition "*things which are equal to the same thing are equal to one another.*" I have therefore, as a rule, omitted in the text all reference to the general statements of axioms, and have only introduced such a statement occasionally, where its introduction seemed to me the shortest way of explaining the nature of the next step in the demonstration.

If it be objected that all axioms used should be clearly stated, and that their number should not be unnecessarily extended, my reply is that neither the Greek text nor any edition of it, with which I am acquainted, has attempted to make its list of axioms perfect in either of these respects. The lists err in excess, inasmuch as some of the axioms therein can be deduced from others: they err in defect, inasmuch as in the demonstrations of Propositions conclusions are often drawn, to support the validity of which no appeal can be made to any axiom in the lists.

Under the term *postulate* I have included not only what may be called the postulates of geometrical operation, such as "*it is assumed that a straight line may be drawn from any point to any other point,*" but also geometrical theorems, the truth of which we assume, such as "*two straight lines cannot have a common part.*"

The postulates of this edition are nine in number.

Postulates 3, 4, 6 are the postulates of geometrical operation, which are common to all editions of the Elements of Euclid. Postulates 1, 5, 9 are the Axioms 10, 11, 12 of modern editions. Postulates 2, 7, 8 do not appear under the head either of axioms or of postulates in Euclid's text, but the substance of them is assumed in the demonstrations of his propositions.

Postulate 9 has been postponed until page 51, as it seemed undesirable to trouble the student with an attempt to unravel its meaning, until he was prepared to accept it as the converse of a theorem, with the proof of which he had already been made acquainted.

It may be mentioned that a proof of Postulate 5, "*all right angles are equal*" is given in the text (Proposition 10 B), and that therefore the number of the Postulates might have been diminished by one: it was however thought necessary to retain this Postulate in the list, so that it might be used as a postulate by any person who might prefer to adhere closely to the original text of Euclid.

One important feature in the present edition is the greater freedom in the direct use of "the method of superposition" in the proofs of the Propositions. The method is used directly by Euclid in his proof of Proposition 4 of Book I., and indirectly in his proofs of Proposition 5 and of every other Proposition, in which the theorem of Proposition 4 is quoted. It seems therefore but a slight alteration to adopt the direct use of this method in the proofs of any theorems, in the proofs of which, in Euclid's text, the theorem of Proposition 4 is quoted.

It may of course be fairly objected that it would be more logical for a writer, who uses with freedom the method of superposition, to omit the first three Propositions of Book I. To this objection my reply must be that it is considered undesirable to alter the numbering of the Propositions in Books I. and II. at all events. No doubt a work written merely for the teaching of geometry, without immediate reference to the requirements of candidates preparing for examination, might well omit the first three Propositions and assume as a postulate that "*a circle may be described with any point as centre, and with a length equal to any given straight line as radius,*" instead of the postulate of Euclid's text (Postulate 6 of the present edition), "*a circle may be described with any point as centre and with any straight line drawn from that point as radius.*"

The use of the words "each to each" has been abandoned. The statement that two things are equal to two other things each to each, seems to imply, according to the natural meaning of the words, that all four things are equal to each other. Where we wish to state briefly that A has a certain relation to a, B has the same relation to b, and C has the same relation to c, we prefer to say that A, B, C have this relation to a, b, c **respectively.**

The enunciations of the Propositions in Books I. and II. have been, with some few slight exceptions, retained throughout, and the order of the Propositions remains unaltered, but different methods of proof have been adopted in many cases. The chief instances of alteration are to be found in Propositions 5 and 6 of Book I., and in Book II.

The use of what may be called impossible figures, such as occurred in Euclid's text in the proofs of Propositions

6 and 7 of Book I. has been avoided. It seems better to prove that a line cannot be drawn satisfying a certain condition without making a pretence of doing what is impossible.

Two Propositions (10 A and 10 B), have been introduced to shew that, if the method of superposition be used, we need not take as a postulate "*all right angles are equal to one another*," but that we may deduce this theorem from other postulates which have been already assumed.

Another new Proposition introduced into the text is Proposition 26 A, "*if two triangles have two sides equal to two sides, and the angles opposite to one pair of equal sides equal, the angles opposite to the other pair are either equal or supplementary*," which may be described, with reference to Euclid's text, as the missing case of the equality of two triangles. It is intimately connected with what is called in Trigonometry "the ambiguous case" in the solution of triangles.

Another new Proposition (41 A) is the solution of the problem "*to construct a triangle equal to a given rectilineal figure.*" It appears to be a more practical method of solving the general problem of Proposition 45 "*to construct a parallelogram equal to a given rectilineal figure, having a side equal to a given straight line, and having an angle equal to a given angle*," to begin with the construction of a triangle equal to the given figure rather than to follow the exact sequence of Euclid's propositions.

In the notes a few "Additional Propositions" have been introduced containing important theorems, which did not occur in Euclid's text, but with which it is desirable that the student should become familiar as early as possible. Also outlines have been given of some of the many different proofs which have been discovered of Pythagoras's Theorem. They may be found interesting and useful as exercises for the student.

Euclid's proofs of many of the Propositions of Book II. are unnecessarily long. His use of the diagonal of the square in his constructions in Propositions 4 to 8 can scarcely be considered elegant.

It is curious to notice that Euclid after giving a demonstration of Proposition 1 makes no use whatever of the theorem. It seems more logical to deduce from Proposition 1 those of the subsequent Propositions which can be readily so deduced.

In Book II. outlines of alternative proofs of several of the Propositions have been given, which may be developed more fully and used in examinations, in place of the proofs given in the text. Some of these proofs are not, so far as I know, to be found in English text books. The most interesting ones are those of Propositions 12 and 13. Some, which I thought at first were new, I have since found in foreign text books.

The Propositions in the text have not been distinguished by the words "Theorem" and "Problem." The student may be informed once for all that the word **theorem** is used of a geometrical truth which is to be demonstrated, and that the word **problem** is used of a geometrical construction which is to be performed.

Although Euclid always sums up the result of a Proposition by the words *ὅπερ ἔδει δεῖξαι* or *ὅπερ ἔδει ποιῆσαι*, there seems to be no utility in putting the letters Q.E.D. or Q.E.F. at the end of a Proposition in an English text-book. The words "Quod erat demonstrandum" or "Quod erat faciendum" in a Latin text were not out of place.

When the book is opened, the reader will see as a rule on the left hand page a Proposition, and on the opposite page notes or exercises. The notes are either appropriate to the Proposition they face or introductory to the one next succeeding. The exercises on the right hand page are, it is hoped, in all cases capable of being solved by means of the Proposition on the adjoining page and of preceding Propositions. They have been chosen with care and with the special view of inducing the student from the commencement of his reading to attempt for himself the solution of exercises.

For many Propositions it has been difficult to find suitable exercises: consequently many of the exercises have been specially manufactured for the Propositions to which they are attached. Great pains have been taken to verify the exercises, but notwithstanding it can scarcely be hoped that all trace of error has been eliminated.

It is with pleasure that I record here my deep sense of obligation to many friends, who have aided me by valuable hints and suggestions, and more especially to A. R. Forsyth, M.A., Fellow and Assistant Tutor of Trinity College, Charles Smith, M.A., Fellow and Tutor of Sidney Sussex College, R. T. Wright, M.A., formerly Fellow and Tutor of Christ's College, my brother-in-law the Reverend T. J. Sanderson, M.A., formerly Fellow of Clare College, and my brother W. W. Taylor, M.A., formerly Scholar of Queen's College, Oxford, and afterwards Scholar of Trinity College, Cambridge. The time and trouble ungrudgingly spent by these gentlemen on this edition have saved it from many blemishes, which would otherwise have disfigured its pages.

I shall be grateful for any corrections or criticisms, which may be forwarded to me in connection either with the exercises or with any other part of the work.

H. M. TAYLOR.

TRINITY COLLEGE, CAMBRIDGE,
October 1, 1889.

PREFACE TO BOOKS III AND IV.

IN Book III. the chief deviation from Euclid's text will be found in the first twelve Propositions, where a good deal of rearrangement has been thought desirable. This rearrangement has led to some changes in the sequence of Propositions as well as in the Propositions themselves; but, even with these changes, the first twelve Propositions will be found to include the substance of the whole of the first twelve of Euclid's text.

The Propositions from 13 to 37 are, except in unimportant details, unchanged in substance and in order.

The enunciation of the theorem of Proposition 36 has been altered to make it more closely resemble that of the complementary theorem of Proposition 35.

An additional Proposition has been introduced on page 186 involving the principle of the rotation of a plane figure about a point in its plane. It is a principle of which extensive use might with advantage be made in the proof of some of the simpler properties of the circle. It has not however been thought desirable to do more in this edition than to introduce the student to this method and by a selection of exercises, which can readily be solved by its means, to indicate the importance of the method.

PREFACE.

The elegant theorem commonly called Ptolemy's Theorem is not found in Euclid's text. It was incorporated by Dr Robert Simson in one of his later editions as Proposition D of Book VI. The position in those modern editions which are based on Simson's conveys the idea that the methods of proportion are necessary for the proof of the theorem. I have been bold enough to insert it as Proposition 37 B in Book III, prefacing it by another theorem which appears as Proposition 37 A.

The additional Propositions at the end of Book III will introduce the reader to some of the simpler properties of Poles and Polars, of the Radical Axis of two Circles, of Orthogonal Circles, and of the Nine Point Circle of a Triangle.

Some of the definitions usually found in Book IV have been transferred in this edition to Book III. The change has been made chiefly with a view to convenience in the enunciation of exercises.

There is nothing which calls for special remark in Book IV.

I record with gratitude my obligations to those friends who have again so generously in this the second instalment of my work rendered me every assistance.

H. M. T.

TRINITY COLLEGE, CAMBRIDGE.
January 8, 1891.

PREFACE TO BOOKS V. AND VI.

THE portion of Book V which is included in this edition contains only so much as is sufficient to establish in their generality those theorems in proportion which in Book VI are applied to geometrical magnitudes.

The method commonly used of representing the magnitudes under discussion by straight lines has been discarded as tending to mislead the student with respect to the general nature of the theorems of Book V, which are applicable to magnitudes of many kinds besides those met with in Geometry. In fact Book V is not essentially geometrical; a student whose knowledge of proportion is derived entirely from any of the standard treatises on Algebra is fully prepared to follow the applications of proportion to geometrical magnitudes as developed in Book VI.

In consequence of the omission of some of the Propositions of the Greek text, the Propositions in Book V have been renumbered; but for convenience the numbering of the Greek text is printed in brackets at the head of each Proposition.

In Book VI a slight departure from Euclid's text is made in the treatment of similar figures. The definition of similar polygons which is adopted in this work brings into prominence the important property of the fixed ratio of their corresponding sides. Its use has the great merit of tending at once to simplicity and brevity in the proofs of many theorems.

The numbering of the Propositions in Book VI remains unchanged: Propositions 27, 28, 29 are omitted as in many of the recent English editions of Euclid, and in several cases a Proposition which consists of a theorem and its converse is divided into two Parts. Proposition 32 of Euclid's text, which is a very special case of no great interest, has been replaced by a simple but important theorem in the theory of similar and similarly situate figures.

The chief difficulty with respect to the additions which have been made to Book VI was the immense number of known theorems from which a selection had to be made.

I have attempted by means of two or three series of Propositions arranged in something like logical sequence to introduce the student to important general methods or well-known interesting results.

One series gives a sketch of the theory of transversals, and the properties of harmonic and anharmonic ranges and pencils, and leads up to Pascal's Theorem. Another series deals with similar and similarly situate figures and leads up to Gergonne's elegant solution of the problem to describe a circle to touch three given circles. These are followed by an introduction to the method of Inversion, an account of Casey's extension of Ptolemy's Theorem, some of the important properties of coaxial circles, and Poncelet's Theorems relating to the porisms connected with a series of coaxial circles.

No attempt has been made to represent the very large and still increasing collection of theorems connected with the "Modern Geometry of the Triangle."

I hereby acknowledge the great help I have received in this portion of my work from friends, and especially from Dr Forsyth and from my brother Mr J. H. Taylor. To the latter I am indebted for the Index to Books I—VI, which I hope may prove of some assistance to persons using this edition.

H. M. T.

TRINITY COLLEGE, CAMBRIDGE,
March 16, 1893.

THE ELEMENTS OF GEOMETRY.

BOOK I.

Definition 1. *That which has position but not magnitude is called* **a point.**

The word *point* is used in many different senses. We speak in ordinary language of the point of a pin, of a pen or of a pencil. Any mark made with such a point on paper is of some definite size and is in some definite position. A small mark is often called a spot or a dot. Suppose such a spot to become smaller and smaller; the smaller it becomes the more nearly it resembles a *geometrical point:* but it is only when the spot has become so small that it is on the point of vanishing altogether, i.e. when in fact the spot still has position but has no magnitude, that it answers to the geometrical definition of a point.

A point is generally denoted by a single letter of the alphabet: for instance we speak of *the point A*.

Definition 2. *That which has position and length but neither breadth nor thickness is called* **a line.**

The extremities of a line are points.

The intersections of lines are points.

The word *line* also is used in many different senses in ordinary language, and in most of these senses the main idea suggested is that of *length*. For instance we speak of a line of railway as connecting two distant towns, or of a sounding line as reaching from the bottom of the sea to the surface, and in so speaking we seldom think of the breadth of the railway or of the thickness of the sounding line.

When we speak of a *geometrical line*, we regard merely the length: we exclude the idea of breadth and thickness altogether: in fact we consider that the cross-section of the line is of no size, or in other words that the cross-section is a geometrical point.

If a point move with a continuous motion from one position to another, the path which it describes during the motion is a line.

DEFINITION 3. *That which has position, length and breadth but not thickness is called* **a surface.**

The boundaries of a surface are lines.

The intersections of surfaces are lines.

The word *surface* in ordinary language conveys the idea of extension in two directions: for instance we speak of the surface of the Earth, the surface of the sea, the surface of a sheet of paper. Although in some cases the idea of the thickness or the depth of the thing spoken of may be present in the speaker's mind, yet as a rule no stress is laid on depth or thickness. When we speak of a *geometrical surface* we put aside the idea of depth and thickness altogether. We are told that it takes more than 300,000 sheets of gold leaf to make an inch of thickness; but although the gold leaf is so thin, it must not be regarded as a geometrical surface. In fact each leaf however thin has always two bounding surfaces. The geometrical surface is to be regarded as absolutely devoid of thickness, and no number of surfaces put together would make any thickness whatever.

DEFINITION 4. *That which has position, length, breadth and thickness is called* **a solid.**

The boundaries of solids are surfaces.

DEFINITION 5. *Any combination of points, lines, and surfaces is called* **a figure.**

DEFINITION 6. *A line which lies evenly between points on it is called* **a straight line.**

This is Euclid's definition of a straight line. It cannot be turned to practical use by itself. We supplement the definition, as Euclid did, by making some assumptions the nature of which will be seen hereafter.

POSTULATES. There are a few geometrical propositions so obvious that we take the truth of them for granted, and a few geometrical operations so simple that we assume we may perform them when we please without giving any explanation of the process. The claim we make to use any one of these propositions, or to perform any one of these operations, is called a **postulate.**

POSTULATE 1. *Two straight lines cannot enclose a space.*

This postulate is equivalent to

Two straight lines cannot intersect in more than one point.

POSTULATE 2. *Two straight lines cannot have a common part.*

If two straight lines have two points A, B in common, they must coincide between A and B, since, if they did not, the two straight lines would enclose a space. Again, they must coincide beyond A and B, since, if they did not, the two straight lines would have a common part. Hence we conclude that

Two straight lines, which have two points in common, are coincident throughout their length.

Thus two points on a straight line completely fix the position of the line. Hence we generally denote a straight line by mentioning two points on it, and when the straight line is of finite length, we generally denote it by mentioning the points which are its two extremities.

For instance, if P and Q be two points on a straight line, the line is called the straight line PQ or the straight line QP, or sometimes more shortly PQ or QP: and the straight line which is *terminated* by two points P and Q is called in the same way PQ or QP.

It may be remarked that, when merely the actual length of the straight line is under discussion, we use PQ or QP indifferently: but that, when we wish to consider the direction of the line, we must carefully distinguish between PQ and QP.

POSTULATE 3. *A straight line may be drawn from any point to any other point.*

POSTULATE 4. *A finite straight line may be produced at either extremity to any length.*

The demands made in Postulates 3 and 4 are in practical geometry equivalent to saying that a 'straight edge' may be used for drawing a straight line from one point to another and for producing a straight line to any length.

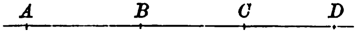
We assume, as Euclid did, that it is possible to shift any geometrical figure from its initial position unchanged in shape and size into another position.

Test of Equality of Geometrical Figures. The criterion of the equality of two geometrical figures, which we shall use in most cases, is the possibility of shifting one of the figures, unchanged in shape and size, so that it exactly fits the place which the other of the figures occupies. (See Def. 21.)

This method of testing the equality of geometrical figures is generally known as **the method of superposition.**

Test of equality of straight lines. Two straight lines AB, CD are said to be equal, when it is possible to shift either of them, say AB, so that it coincides with the other CD, the end A on C and the end B on D, or the end A on D and the end B on C.

Addition of Lines. Having defined the equality of straight lines, we proceed to explain what is meant by the addition of straight lines.



If in a straight line we take points A, B, C, D in order, we say that the straight line AC is the *sum* of the two straight lines AB, BC (or of any two straight lines equal to them),

and that the straight line AB is the *difference* of the two straight lines AC, BC (or of any two straight lines equal to them).

In the same way we say that the straight line AD is the sum of the three straight lines AB, BC, CD.

Again, if AB be equal to BC, we say that AC is double of AB or of BC.

DEFINITION 7. *A surface which lies evenly between straight lines on it is called* **a plane.**

This is Euclid's definition of a plane: there is the same difficulty in making use of it that there is in making use of his definition of a straight line.

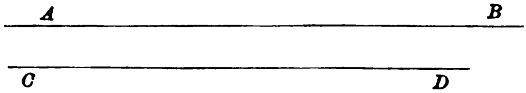
Consequently this definition has by many modern editors been replaced by the following, which perhaps merely expresses Euclid's meaning in other words:

A surface such that the straight line joining any two points in the surface lies wholly in the surface is called **a plane.**

DEFINITION 8. *A figure, which lies wholly in one plane, is called a* **plane figure.**

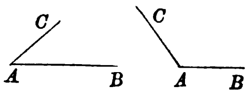
All the geometrical propositions in the first six books of the Elements of Euclid relate to figures in one plane. This part of Geometry is called Plane Geometry.

Definition 9. *Two straight lines in the same plane, which do not meet however far they may be produced both ways, are said to be* **parallel*** **to one another.**



Definition 10. **A plane angle** *is the inclination to one another of two straight lines which meet but are not in the same straight line.*

The idea of an angle is one which it is very difficult to convey by the words of a definition. We will content ourselves by explaining some few things connected with angles.



If two straight lines AB, AC meet at A, the amount of their divergence from one another or their inclination to one another is called *the angle which the lines make with one another* or *the angle between the lines*, or *the angle contained by the lines.*

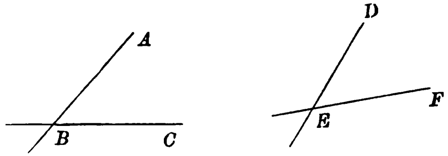
The angle formed by the straight lines AB, AC is generally denominated BAC, or CAB, the middle letter always denoting the point where the lines meet, and the letters B and C denoting *any* two points in the straight lines AB, AC. It must be carefully noted that the magnitude of the angle is not affected by the length of the straight lines AB, AC.

The point A, where the two straight lines AB and AC, which form the angle BAC, meet, is called the **vertex** of the angle BAC.

If there be only two straight lines meeting at a point A, the angle formed by the lines is sometimes denoted by the single letter A.

* Derived from παρά "by the side of" and ἀλλήλας "one another": παράλληλοι γραμμαί "lines side by side".

Test of Equality of Angles. Two angles are said to be equal, when it is possible to shift the straight lines forming one of the angles, unchanged in position relative to each other, so as to exactly coincide in direction with the straight lines forming the other angle.



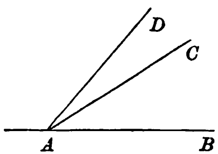
For instance, the angles *ABC*, *DEF* will be equal, if it be possible to shift *AB*, *BC* unchanged in position relative to each other, so that *B* coincides with *E*, and so that also either *BA* coincides in direction with *ED* and *BC* with *EF*, or *BA* coincides in direction with *EF* and *BC* with *ED*.

If a straight line move in a plane, while one point in the line remains fixed, the line is said to turn or revolve about the fixed point. If the revolving line move from any one position to any other position, it generates an angle, and *the amount of turning* from one position to the other is the measure of the magnitude of the angle between the two positions of the line.

For instance each hand of a watch, as long as the watch is going, is turning uniformly round its fixed extremity, and is generating an angle uniformly.

This mode of regarding angles enables us to realize that angles are capable of growing to any size and need not be limited (as in most of the propositions in Euclid's Elements they are supposed to be) to magnitudes less than two right angles. (See Def. 11.)

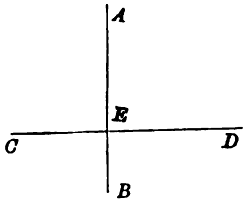
Addition of Angles. If three straight lines *AB*, *AC*, *AD* meet at the same point, we say that the angle *BAD* is the *sum* of the two angles *BAC*, *CAD* (or of any two angles equal to them).



In the same way we say that the angle *BAC* is the *difference* of the two angles *BAD*, *CAD* (or of any two angles equal to them).

Two angles such as *BAC*, *CAD*, which have a common vertex and one common bounding line, are called **adjacent angles**.

DEFINITION 11. *If two adjacent angles made by two straight lines at the point where they meet be equal, each of these angles is called* **a right angle,** *and the straight lines are said to be* **at right angles to each other.**



Either of two straight lines which are at right angles to each other is said to be **perpendicular** *to the other.*

If a straight line *AE* be drawn from a point *A* at right angles to a given straight line *CD*, the part *AE* intercepted between the point and the straight line is commonly called **the perpendicular** from the point *A* on the straight line *CD*.

Euclid uses as a postulate,

POSTULATE 5. *All right angles are equal to one another.*

It is not necessary to assume this proposition, since it can be proved by the method of superposition. A proof will be found on a subsequent page. (p. 37)

DEFINITION 12. *An angle less than a right angle is called* **an acute angle.**

An angle greater than a right angle and less than two right angles is called **an obtuse angle.**

DEFINITION 13. *A line, which is such that it can be described by a moving point starting from any point of the line and returning to it again, is called* **a closed line.**

A figure composed wholly of straight lines is called **a rectilineal figure.**

The straight lines, which form a closed rectilineal figure, are called **the sides** *of the figure.*

The sum of the lengths of the sides of any figure is called **the perimeter** *of the figure.*

The point, where two adjacent sides meet, is called **a vertex** *or* **an angular point** *of the figure.*

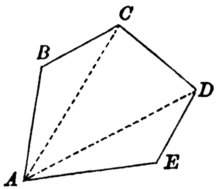
The angle formed by two adjacent sides is called **an angle** *of the figure.*

A straight line joining any two vertices of a closed rectilineal figure, which are not extremities of the same side, is called **a diagonal***.

The surface contained within a closed figure is called **the area** *of the figure.*

A closed rectilineal figure, which is such that the whole figure lies on one side of each of the sides of the figure, is called **a convex figure.**

A closed rectilineal figure is in general denoted by naming the letters, which denote its vertices, in order: for instance the five-sided figure in the diagram is denoted by the letters *A*, *B*, *C*, *D*, *E*, in order: i.e. it might be called the figure *ABCDE*, or the figure *CBAED*.



A, *B*, *C*, *D*, *E* are its *vertices.*

AB, *BC*, *CD*, *DE*, *EA* are its *sides.*

ABC, *BCD*, *CDE*, *DEA*, *EAB* are its *angles.*

AC, *AD* are two of its *diagonals.*

It will be observed that a closed figure has the same number of angles as it has sides.

If a closed figure have an even number of sides, we speak of a pair of sides as being opposite, and of a pair of angles as being opposite.

If a closed figure have an odd number of sides, we speak of an angle as being opposite to a side and *vice versa.*

For instance in the quadrilateral *ABCD* the side *AD* is said to be opposite to the side *BC*, and the angle *BAD* opposite to the angle *BCD*, but in the five-sided figure *ABCDE* the side *CD* is said to be opposite to the angle *BAE*, and the angle *AED* opposite to the side *BC*.

* Derived from διά "through", and γωνία "an angle".

DEFINITION 14. *A figure, all the sides of which are equal, is called* **equilateral.**

A figure, all the angles of which are equal, is called **equiangular.**

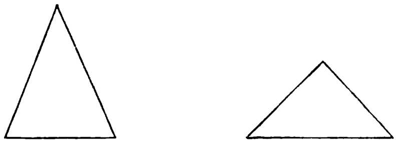
A figure, which is both equilateral and equiangular, is called **regular.**

DEFINITION 15. *A closed rectilineal figure, which has three sides*, is called* **a triangle.**

A closed rectilineal figure, which has four sides, is called **a quadrilateral.**

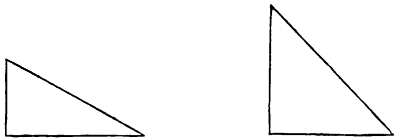
A closed rectilineal figure, which has more than four sides, is called **a polygon**†.

DEFINITION 16. *A triangle, which has two sides equal, is called* **isosceles**‡.



A triangle, which has a right angle, is called **right-angled.**

The side opposite to the right angle is called **the hypotenuse**§.



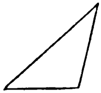
* A figure, which has three sides, must also have three angles. It is for this reason called a *triangle*.

† Derived from πολύς "much" and γωνία "an angle".

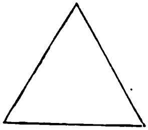
‡ Derived from ἴσος "equal" and σκέλος "a leg".

§ Derived from ὑπό "under" and τείνειν "to stretch". ἡ ὑποτείνουσα γραμμή "the line subtending" or "stretching across" (the right angle).

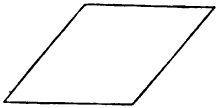
A triangle, which has an obtuse angle, is called **obtuse-angled.**



A triangle, which has three acute angles, is called **acute-angled.**

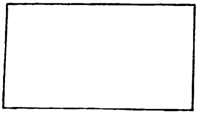


DEFINITION 17. *A quadrilateral, which has four sides equal, is called* **a rhombus.**



DEFINITION 18. *A quadrilateral, whose opposite sides are parallel, is called* **a parallelogram.**

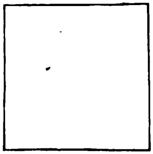
DEFINITION 19. *A parallelogram, one of whose angles is a right angle, is called* **a rectangle.**



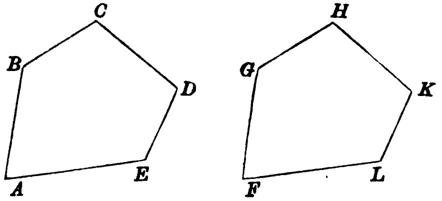
It will be proved later that each angle of a rectangle is a right angle.

DEFINITION 20. *A rectangle, which has two adjacent sides equal, is called* **a square.**

It will be proved later that all the sides of a square are equal.

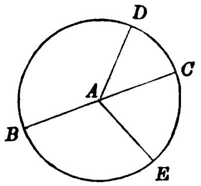


DEFINITION 21. *Two figures are said to be* **equal in all respects,** *when it is possible to shift one unchanged in shape and size so as to coincide with the other.*



The figures $ABCDE$, $FGHKL$ are equal in all respects, if it be possible to shift $ABCDE$ so that the vertices A, B, C, D, E may coincide with the vertices F, G, H, K, L respectively: in which case the sides of the two figures must be equal, AB, BC, CD, DE, EA to FG, GH, HK, KL, LF respectively, and the angles must be equal, ABC, BCD, CDE, DEA, EAB to FGH, GHK, HKL, KLF, LFG respectively.

DEFINITION 22. *A plane closed line, which is such that all straight lines drawn to it from a fixed point are equal, is called* **a circle.**



This point is called **the centre** *of the circle.*

It will be proved hereafter that a circle has only one centre.

A straight line drawn from the centre of the circle to the circle is called **a radius.**

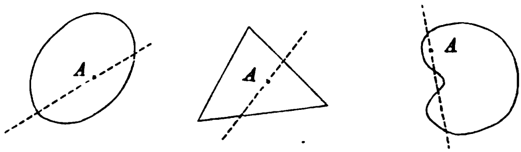
A straight line drawn through the centre and terminated both ways by the circle is called **a diameter.**

It will be proved hereafter that three points on a circle completely fix the position and magnitude of the circle: hence we generally denote a circle by mentioning three points on it; for instance the circle in the diagram might be called the circle BDE, or the circle DBC.

The one assumption which we make with reference to describing a circle is contained in the following postulate:

POSTULATE 6. *A circle may be described with any point as centre and with any straight line drawn from that point as radius.*

POSTULATE 7. *Any straight line drawn through a point within a closed figure must, if produced far enough, intersect the figure in two points at least.*



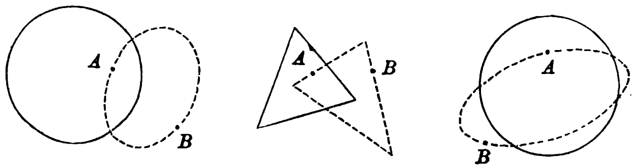
In the diagram we have three specimens of closed figures each with a point A inside the figure.

It is easily seen that any straight line through A must intersect the figure in two points at least: in the case of two of the figures a straight line cannot intersect the figure in more than two points: but in the third case, a straight line can be drawn to intersect the figure in four points.

POSTULATE 8. *Any line joining two points one within and the other without a closed figure must intersect the figure in one point at least.*

It follows that

Any closed line drawn through two points one within and the other without a closed figure must intersect the figure in two points at least.



In the diagram we have three specimens of closed figures with two points A, B, one inside and the other outside the figure.

It is easily seen that any line joining A and B must intersect the figure in one point at least, and that any closed line drawn through A and B must intersect the figure in two points at least: in two of the cases in the diagram either of the paths represented by part of the dotted line joining A and B intersects the figure in one point only and the closed line drawn intersects the figure in two points only: but in the third case one of the paths from A to B represented by part of the dotted line intersects the figure in one point only, while the path represented by the other part of the dotted line intersects the figure in three points, and the closed line drawn through A and B intersects the figure in four points.

AXIOMS. There are a number of simple propositions generally admitted to be true universally, i.e. with reference to magnitudes of all kinds.

Such propositions were called by Euclid κοιναὶ ἔννοιαι, "common notions": they are now usually denominated *axioms**, ἀξιώματα, as being propositions claimed without demonstration.

The following are examples of such axioms:

Things which are equal to the same thing are equal to one another.

If equals be added to equals, the wholes are equal.

If equals be taken from equals, the remainders are equal.

Doubles of equals are equal.

Halves of equals are equal.

The whole of a thing is greater than a part.

If one thing be greater than a second and the second greater than a third, the first is greater than the third.

Such propositions as the above we shall use freely in the following pages without further remark.

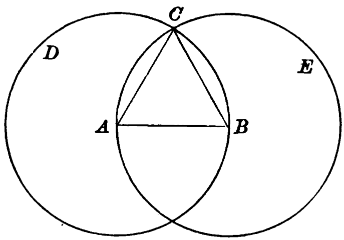
* Dr Johnson in his English Dictionary defined an axiom as "a proposition evident at first sight, that cannot be made plainer by demonstration."

PROPOSITION 1.

On a given finite straight line to construct an equilateral triangle.

Let AB be the given finite straight line:
it is required to construct an equilateral triangle on AB.

CONSTRUCTION. With A as centre and AB as radius, describe the circle BCD. (Post. 6.)
With B as centre and BA as radius, describe the circle ACE.
These circles must intersect: (Post. 8.)
let them intersect in C.
Draw the straight lines CA, CB: (Post. 3.)
then ABC is a triangle constructed as required.



PROOF. Because A is the centre of the circle BCD,
AC is equal to AB. (Def. 22.)
And because B is the centre of the circle ACE,
BC is equal to BA.
Therefore CA, AB, BC are all equal.

Wherefore, *the triangle ABC is equilateral, and it has been constructed on the given finite straight line AB.*

It is assumed in this proposition that the two circles intersect. It is easily seen that they must intersect in *two* points. We can take either of these points as the third angular point of an equilateral triangle on the given straight line; there are thus *two* triangles which can be constructed satisfying the requirements of the proposition. We say therefore that the problem put before us in this proposition admits of two solutions.

We shall often have occasion to notice that a geometrical problem admits of more than one solution, and it is a very useful exercise to consider the number of possible solutions of a particular problem.

For the future we shall generally use the abbreviated expression "draw AB" instead of "draw the straight line AB" or "draw a straight line from the point A to the point B."

EXERCISES.

1. Produce a straight line so as to be (*a*) twice, (*b*) three times, (*c*) five times, its original length.

2. Construct on a given straight line an isosceles triangle, such that each of its equal sides shall be (*a*) twice, (*b*) three times, (*c*) six times, the length of the given line.

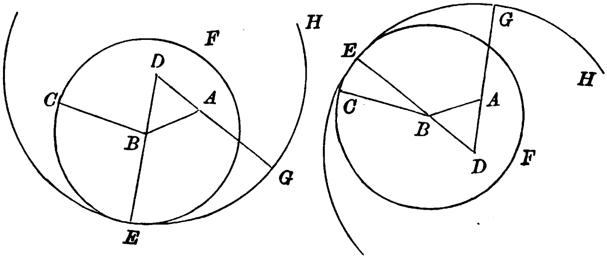
3. Prove that, if two circles, whose centres are A, B, and whose radii are equal, intersect in C, D, the figure $ABCD$ is a rhombus.

PROPOSITION 2.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line: it is required to draw from A a straight line equal to BC.

CONSTRUCTION. Draw AB, the straight line from A to one of the extremities of BC; (Post. 3.)
on it construct an equilateral triangle ABD. (Prop. 1.)
With B as centre and BC as radius, describe the circle CEF, (Post. 6.)
meeting DB (produced if necessary) at E. (Post. 7.)
With D as centre and DE as radius, describe the circle EGH,
meeting DA (produced if necessary) at G:
then AG is a straight line drawn as required.



PROOF. Because B is the centre of the circle CEF,
BC is equal to BE. (Def. 22.)
Again, because D is the centre of the circle EGH,
DG is equal to DE;
and because ABD is an equilateral triangle,
DA is equal to DB; (Def. 14.)
therefore AG is equal to BE.
And it has been proved that BC is equal to BE;
therefore AG is equal to BC.

Wherefore, *from the given point A a straight line AG has been drawn equal to the given straight line BC.*

It is assumed in this proposition that the straight line DB intersects the circle CEF. It is easily seen that it must intersect it *in two points*.

It will be noticed that in the construction of this proposition there are several steps at which a choice of two alternatives is afforded: (1) we can draw either AB or AC as the straight line on which to construct an equilateral triangle: (2) we can construct an equilateral triangle on either side of AB: (3) if DB cut the circle in E and I, we can choose either DE or DI as the radius of the circle which we describe with D as centre.

There are therefore three steps in the construction, at each of which there is a choice of two alternatives: the total number of solutions of the problem is therefore $2 \times 2 \times 2$ or eight.

On the opposite page two diagrams are drawn, to represent two out of these eight possible solutions. It will be a useful exercise for the student to draw diagrams corresponding to some of the remaining six.

EXERCISES.

1. Draw a diagram for the case in which the given point is the middle point of the given straight line.

2. Draw a diagram for the case in which the given point is in the given straight line produced.

3. Draw from a given point a straight line (*a*) twice, (*b*) three times the length of a given straight line.

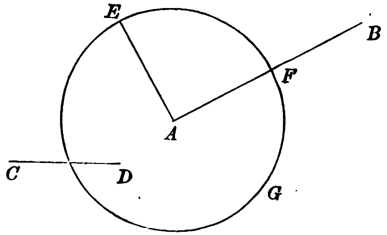
4. Draw from D in any one of the diagrams of Proposition 2 a straight line, so that the part of it intercepted between the two circles may be equal to the given straight line. Is a solution always possible?

PROPOSITION 3.

From the greater of two given straight lines to cut off a part equal to the less.

Let AB and CD be the two given straight lines, of which AB is the greater:
it is required to cut off from AB a part equal to CD.

CONSTRUCTION. From A draw a straight line AE equal to CD; (Prop. 2.)
with A as centre and AE as radius,
describe the circle EFG. (Post. 6.)
The circle must intersect AB between A and B,
for AB is greater than AE.
Let F be the point of intersection:
then AF is the part required.



PROOF. Because A is the centre of the circle EFG,
AE is equal to AF. (Def. 22.)
But AE was made equal to CD; (Construction.)
therefore AF is equal to CD.

Wherefore, *from AB the greater of two given straight lines a part AF has been cut off equal to CD the less.*

The demand made in Postulate 6, that "a circle may be described with any point as centre and with any straight line drawn from that point as radius," is equivalent, in practical geometry, to saying that a pair of compasses may be used in the following manner: the extremity of one leg of a pair of compasses may be put down on any point *A*, the compasses may then be opened so that the extremity of the other leg comes to any other point and then a circle may be swept out by the extremity of the second leg of the compasses, the extremity of the first leg remaining throughout the motion on the point *A*.

Compasses are also used practically for carrying a given length from any one position to any other: for instance, they would generally be used to solve the problem of Proposition 3 by opening the compasses out till the extremities of the legs came to the points *C*, *D*: they would then be shifted, without any change in the opening of the legs, until the extremity of one leg was on *A* and the extremity of the other in the straight line *AB*.

Euclid restricted himself much in the same way as a draughtsman would, if he allowed himself only the first mentioned use of the compasses: the first three propositions shew how Euclid with this self-imposed restriction solved the problem, which without such a restriction could have been solved more readily.

After the problems in the first three propositions have been solved, we may assume that we can draw a circle, as a practical draughtsman would, with any point as centre and with a length equal to any given straight line as radius.

EXERCISES.

1. On a given straight line describe an isosceles triangle having each of the equal sides equal to a second given straight line.

2. Construct upon a given straight line an isosceles triangle having each of the equal sides double of a second given straight line.

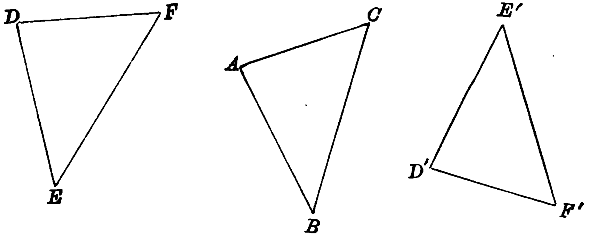
3. Construct a rhombus having a given angle for one of its angles, and having its sides each equal to a given straight line.

PROPOSITION 4.

If two triangles have two sides of the one equal to two sides of the other, and also the angles contained by those sides equal, the two triangles are equal in all respects.

(See Def. 21.)

Let ABC, DEF be two triangles, in which AB is equal to DE, and AC to DF, and the angle BAC is equal to the angle EDF:
it is required to prove that the triangles ABC, DEF are equal in all respects.



Proof. Because the angles BAC, EDF are equal,
it is possible to shift the triangle ABC
so that A coincides with D,
and AB coincides in direction with DE,
and AC with DF. (Test of Equality, page 8.)

If this be done,
because AB is equal to DE,
B must coincide with E;
and because AC is equal to DF,
C must coincide with F.

Again because B coincides with E and C with F,
BC coincides with EF; (Post. 2.)
therefore the triangle ABC coincides with the triangle DEF,
and is equal to it in all respects.

Wherefore, *if two triangles* &c.

The proof of this proposition holds good not only for a pair of triangles such as *ABC*, *DEF* in the diagram: it holds good equally for a pair such as *ABC*, *D'E'F'*, one of which must be reversed or turned over before the triangles can be made to coincide or fit exactly.

In this proposition Euclid assumed Postulate 2, that two straight lines cannot have a common part. When the triangle *ABC* is shifted, so that *A* is on *D* and *AB* is on *DE*, there would be no justification for the conclusion that *B* must coincide with *E*, because *AB* is equal to *DE*, if it were possible for two straight lines to have a common part. In fact, two curved lines might be drawn from the point *D* starting in the same direction *DE* but leading to two totally distinct points *E* and *F* although the lines were of the same length. It is tacitly assumed, that if the lines be straight lines, this is impossible.

EXERCISES.

1. If the straight line joining the middle points of two opposite sides of a quadrilateral be at right angles to each of these sides, the other two sides are equal.

2. If in a quadrilateral *ABCD* the sides *AB*, *CD* be equal and the angles *ABC*, *BCD* be equal, the diagonals *AC*, *BD* are equal.

3. If in a quadrilateral two opposite sides be equal, and the angles which a third side makes with the equal sides be equal, the other angles are equal.

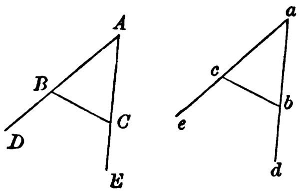
4. Prove by the method of superposition that, if in two quadrilaterals *ABCD*, *A'B'C'D'*, the sides *AB*, *BC*, *CD* be equal to the sides *A'B'*, *B'C'*, *C'D'* respectively, and the angles *ABC*, *BCD* equal to the angles *A'B'C'*, *B'C'D'* respectively, the quadrilaterals are equal in all respects.

PROPOSITION 5.

If two sides of a triangle be equal, the angles opposite to these sides are equal, and the angles made by producing these sides beyond the third side are equal.

Let *ABC* be a triangle, in which *AB* is equal to *AC*, and *AB*, *AC* are produced to *D*, *E*:
it is required to prove that the angle *ACB* is equal to the angle *ABC*, and the angle *BCE* to the angle *CBD*.

Construction. Let the figure *ABCDE* be turned over and shifted unchanged in shape and size to the position *abcde*, *A* to *a*, *B* to *b*, *C* to *c*, *D* to *d* and *E* to *e*.



Proof. Because the angles *DAE*, *ead* are equal, it is possible to shift the figure *abcde*
so that *a* coincides with *A*,
and *ae* coincides in direction with *AD*,
and *ad* with *AE*. (Test of Equality, page 8.)

If this be done,
because *ac* is equal to *AB*,
c must coincide with *B*;
and because *ab* is equal to *AC*,
b must coincide with *C*;
hence *cb* coincides with *BC*. (Post. 2.)

Now because *ace* coincides in direction with *ABD*,
and *cb* with *BC*,
the angle *acb* coincides with the angle *ABC*,
and the angle *bce* with the angle *CBD*;
therefore the angle *acb* is equal to the angle *ABC*,
and the angle *bce* to the angle *CBD*.

But the angle *acb* is equal to the angle *ACB*,
and the angle *bce* to the angle *BCE*;
therefore the angle *ACB* is equal to the angle *ABC*,
and the angle *BCE* to the angle *CBD*.

Wherefore, *if two sides* &c.

COROLLARY 1. *An equilateral triangle is also equiangular.*

COROLLARY 2. *If two angles of a triangle be unequal, the sides opposite to these angles are unequal.*

EXERCISES.

1. The opposite angles of a rhombus are equal.

2. If a quadrilateral have two pairs of equal adjacent sides, it has one pair of opposite angles equal.

3. If in a quadrilateral *ABCD*, *AB* be equal to *AD* and *BC* to *DC*, the diagonal *AC* bisects each of the angles *BAD*, *BCD*.

4. If in a quadrilateral *ABCD*, *AB* be equal to *AD* and *BC* to *DC*, the diagonal *BD* is bisected at right angles by the diagonal *AC*.

5. Prove that the triangle, whose vertices are the middle points of the sides of an equilateral triangle, is equilateral.

6. Prove that the triangle, formed by joining the middle points of the sides of an isosceles triangle, is isosceles.

7. Prove by the method of superposition that, if in a convex quadrilateral *ABCD*, *AB* be equal to *CD* and the angle *ABC* to the angle *BCD*, *AD* is parallel to *BC*.

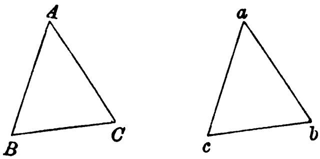
PROPOSITION 6.

If two angles of a triangle be equal, the sides opposite to these angles are equal.

Let ABC be a triangle, in which the angle ABC is equal to the angle ACB:

it is required to prove that AC is equal to AB.

CONSTRUCTION. Let the triangle ABC be turned over and shifted unchanged in shape and size to the position abc, A to a, B to b, and C to c.



PROOF. Because the sides BC, cb are equal,
it is possible to shift the triangle acb
so that cb coincides with BC, c with B, and b with C,
(Test of Equality, page 5.)
and so that the triangles acb, ABC are on the same side of BC.

If this be done,
because the angles ABC, acb are equal,
ca must coincide in direction with BA;
and because the angles ACB, abc are equal,
ba must coincide in direction with CA.

And because two straight lines cannot intersect in more than one point, (Post. 1.)
the point a, which is the intersection of ca and ba, must coincide with the point A, which is the intersection of BA and CA.

Now because a coincides with A and c with B,
ac coincides with AB and is equal to it.
But ac is equal to AC;
therefore AC is equal to AB.

Wherefore *if two angles* &c.

COROLLARY. *An equiangular triangle is also equilateral.*

When in two propositions the hypothesis of each is the conclusion of the other, each proposition is said to be **the converse** of the other.

The theorems in Propositions 5 and 6 are the converses of each other.

It must not be assumed that the converse of a proposition is necessarily true.

EXERCISES.

1. Shew that, if the angles ABC and ACB at the base of an isosceles triangle be bisected by the straight lines BD and CD, DBC will be an isosceles triangle.

2. BAC is a triangle having the angle B double of the angle A. If BD bisect the angle B and meet AC at D, BD is equal to AD.

3. Prove by the method of superposition that, if in two triangles ABC, $A'B'C'$ the angles ABC, BCA be equal to the angles $A'B'C'$, $B'C'A'$ respectively and the sides BC, $B'C'$ be equal, the triangles are equal in all respects.

4. Prove by the method of superposition that, if in two quadrilaterals $ABCD$, $A'B'C'D'$ the angles DAB, ABC, BCD be equal to the angles $D'A'B'$, $A'B'C'$, $B'C'D'$ respectively, and the sides AB, BC be equal to the sides $A'B'$, $B'C'$ respectively, the quadrilaterals are equal in all respects.

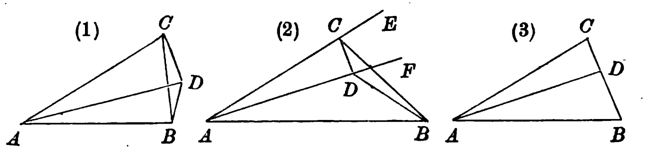
5. If in a quadrilateral $ABCD$, AB be equal to AD and the angle ABC to the angle ADC, then BC is equal to DC, and the diagonal AC bisects the quadrilateral and two of its angles.

PROPOSITION 7.

If two points on the same side of a straight line be equidistant from one point in the line, they cannot be equidistant from any other point in the line.

Let AB be a given straight line, and C, D be two points on the same side of it equidistant from the point A: it is required to prove that C, D cannot be equidistant from any other point in the line.

CONSTRUCTION. Take any other point B in the line, and draw BC, BD.



PROOF. Because C and D are two different points, either

(1) the vertex of each of the triangles ABC, ABD must be outside the other triangle,

(2) the vertex of one triangle must be inside the other,

or (3) the vertex of one triangle must be on a side of the other.

(1) First let the vertex of each triangle be without the other.

Because AD is equal to AC,
the angle ACD is equal to the angle ADC. (Prop. 5.)
But the angle ACD is greater than the angle BCD,
and the angle BDC is greater than the angle ADC:
therefore the angle BDC is greater than the angle BCD;
therefore BC, BD are unequal. (Prop. 5, Coroll. 2.)

(2) Next let the vertex D of one triangle ABD be within the other triangle ABC:

produce AC, AD to E, F. (Post. 4.)

Then because in the triangle *ACD*, *AC* is equal to *AD*, the angles *ECD*, *FDC* made by producing the sides *AC*, *AD* are equal. (Prop. 5.)

But the angle *ECD* is greater than the angle *BCD*,
and the angle *BDC* is greater than the angle *FDC*;
therefore the angle *BDC* is greater than the angle *BCD*;
therefore *BC*, *BD* are unequal. (Prop. 5, Coroll. 2.)

(3) Next let the vertex *D* of one triangle lie on one of the sides *BC* of the other:

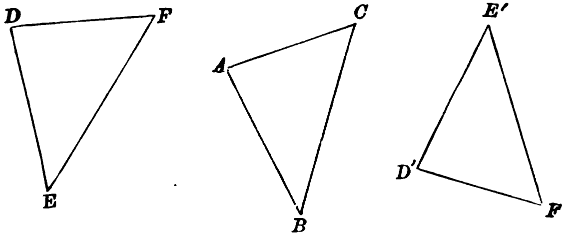
then *BC*, *BD* are unequal.

Wherefore, *if two points on the same side* &c.

PROPOSITION 8.

If two triangles have three sides of the one equal to three sides of the other, the triangles are equal in all respects.

Let *ABC*, *DEF* be two triangles, in which *AB* is equal to *DE*, *AC* to *DF*, and *BC* to *EF*:
it is required to prove that the triangles *ABC*, *DEF* are equal in all respects.



PROOF. Because the sides *BC*, *EF* are equal,
it is possible to shift the triangle *ABC*,
so that *BC* coincides with *EF*, *B* with *E* and *C* with *F*,
(Test of Equality, page 5.)
and so that the triangles *ABC*, *DEF* are on the same side of *EF*.
If this be done,
A must coincide with *D*:
for there cannot be two points on the same side of the straight line *EF* equidistant from *E*
and also equidistant from *F*. (Prop. 7.)
Now because *A* coincides with *D*,
and *B* coincides with *E*, (Constr.)
AB coincides with *DE*. (Post. 2.)
Similarly it can be proved that
AC coincides with *DF*.
Therefore the triangle *ABC* coincides with the triangle *DEF*,
and is equal to it in all respects.

Wherefore, *if two triangles* &c.

EXERCISES.

1. If a quadrilateral have two pairs of equal sides, it must have one pair and may have two pairs of equal angles.

2. *ABC*, *DBC* are two isosceles triangles on the same base *BC*, and on the same side of it: shew that *AD* bisects the vertical angles of the triangles.

3. If the opposite sides of a quadrilateral be equal, the opposite angles are equal.

4. If in a quadrilateral two opposite sides be equal, and the diagonals be equal, the quadrilateral has two pairs of equal angles.

5. If in a quadrilateral the sides *AB*, *CD* be equal and the angles *ABC*, *BCD* be equal, the angles *CDA*, *DAB* are equal.

6. The sides *AB*, *AD* of a quadrilateral *ABCD* are equal, and the diagonal *AC* bisects the angle *BAD*; shew that the sides *CB*, *CD* are equal, and that the diagonal *AC* bisects the angle *BCD*.

7. *ACB*, *ADB* are two triangles on the same side of *AB*, such that *AC* is equal to *BD*, and *AD* is equal to *BC*, and *AD* and *BC* intersect at *O*: shew that the triangle *AOB* is isosceles.

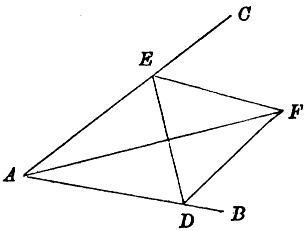
8. A diagonal of a rhombus bisects each of the angles through which it passes.

PROPOSITION 9.

To bisect a given angle.

Let BAC be the given angle:
it is required to bisect it.

CONSTRUCTION. Take any point D in AB,
and from AC cut off AE equal to AD. (Prop. 3.)
Draw DE,
and on DE, on the side away from A, construct the equilateral triangle DEF. (Prop. 1.)
Draw AF:
then AF is the required bisector of the angle BAC.



PROOF. Because in the triangles DAF, EAF,
DA is equal to EA,
AF to AF,
and FD to FE,
the triangles are equal in all respects; (Prop. 8.)
therefore the angle DAF is equal to the angle EAF.

Wherefore, *the given angle BAC is bisected by the straight line AF.*

In the construction for this proposition it is said that the equilateral triangle DEF is to be constructed on the side of DE *away from* A. This restriction is introduced in order to prevent the possibility of the point F coinciding with the point A.

In practical geometry it is always desirable to obtain two points, which determine a straight line, as far apart as possible, as then an error in the position of one of the points causes less error in the position of the straight line.

EXERCISES.

1. A straight line, bisecting the angle contained by two equal sides of a triangle, bisects the third side.

2. The bisectors of the angles ABC, ACB of a triangle ABC meet in D: prove that, if DB, DC be equal, AB, AC are equal.

3. Prove that there is only one bisector of a given angle.

4. Prove that the bisectors of the angles of an equilateral triangle meet in a point.

5. Prove that the bisectors of the angles of an isosceles triangle meet in a point.

6. BAC is a given angle; cut off AB, AC equal to one another: with centres B, C describe circles having equal radii: if the circles intersect at D, AD bisects the angle BAC.

7. Prove by the method of superposition that, if one diagonal of a quadrilateral bisect each of the angles through which it passes, the two diagonals are at right angles to each other.

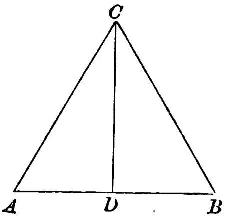
PROPOSITION 10.

To bisect a given finite straight line.

Let AB be the given finite straight line:
it is required to bisect it.

Construction. On AB construct an equilateral triangle ABC, (Prop. 1.)
and bisect the angle ACB by the straight line CD, meeting AB at D: (Prop. 9.)

then AB is bisected as required at D.



Proof. Because in the triangles ACD, BCD,
AC is equal to BC,
and CD to CD,
and the angle ACD is equal to the angle BCD,
the triangles are equal in all respects; (Prop. 4.)
therefore AD is equal to BD.

Wherefore, *the given finite straight line AB is bisected at the point D.*

EXERCISES.

1. Prove that there is only one point of bisection of a given finite straight line.

2. If two circles intersect, then the straight line joining their centres bisects at right angles the straight line joining their points of intersection.

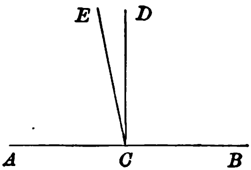
3. Draw from the vertex of a triangle to the opposite side a straight line, which shall exceed the smaller of the other sides as much as it is exceeded by the greater.

PROPOSITION 10 A.

From the same point in a given straight line and on the same side of it, only one straight line can be drawn at right angles to the given straight line.

From the point C in the straight line AB let the straight line CD be drawn at right angles to AB:
it is required to prove that no other straight line can be drawn from C at right angles to AB on the same side of it.

CONSTRUCTION. Draw from C within the angle ACD any straight line CE.



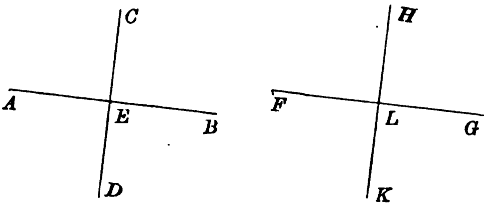
PROOF. Because the angles DCB, DCA are equal,
and the angle ECB is greater than the angle DCB,
and the angle DCA is greater than the angle ECA;
therefore the angle ECB is greater than the angle ECA;
therefore CE is not at right angles to AB. (Def. 11.)
Similarly it can be proved that no straight line drawn from C within the angle DCB can be at right angles to AB.
Therefore no straight line other than CD drawn from C can be at right angles to AB on the same side of it.

Wherefore, *from the same point* &c.

PROPOSITION 10 B.

All right angles are equal to one another.

Let the straight lines *AB*, *CD* meet at *E* and make the angles *CEA*, *CEB* right angles; (Def. 11.)
and let the straight lines *FG*, *HK* meet at *L* and make the angles *HLF*, *HLG* right angles:
it is required to prove that the angle *CEA* is equal to the angle *HLF*.



Proof. If the figure *ABCDE* be shifted, so that *E* coincides with *L*, and the line *AB* in direction with *FG*, and so that *EC*, *LH* are on the same side of *FG*:
then *EC* must coincide with *LH*, for at the same point *L* in *FG* on the same side of it there cannot be two straight lines at right angles to *FG*; (Prop. 10 A.)
therefore the angle *AEC* coincides with the angle *FLH*, and is equal to it.

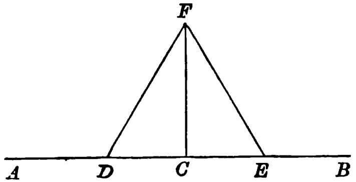
Wherefore, *all right angles* &c.

PROPOSITION 11.

To draw a straight line at right angles to a given straight line from a given point in it.

Let AB be the given straight line, and C the given point in it:
it is required to draw from C a straight line at right angles to AB.

CONSTRUCTION. Take any point D in AC,
and from CB cut off CE equal to CD. (Prop. 3.)
On DE construct an equilateral triangle DFE, (Prop. 1.)
and draw CF:
then CF is a straight line drawn as required.

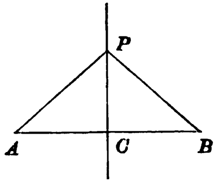


PROOF. Because in the triangles DCF, ECF,
DC is equal to EC,
CF to CF,
and FD to FE, (Constr.)
the triangles are equal in all respects; (Prop. 8.)
therefore the angle DCF is equal to the angle ECF,
and they are adjacent angles;
therefore each of these angles is a right angle; and the straight lines are at right angles to each other. (Def. 11.)

Wherefore, *CF has been drawn at right angles to the given straight line AB, from the given point C in it.*

In the definition of a circle (Def. 22) we meet with the idea of a point, which moves subject to a given condition, the condition being that the point is always to be at a given distance from a given point, i.e. from the centre of the circle. The *path* of such a moving point, or the place (*locus*), at some position on which the point must always be and at any position on which the point may be, is called the *locus* of the point. Hence we say in the case just mentioned that the locus of a point which is at a given distance from a given point is a circle.

As a further illustration of this idea, let us consider the locus of a point, which moves subject to the condition that it is always to be equidistant from two given points.



Let A, B be two given points, and let P be a point equidistant from A and B, i.e. let PA be equal to PB.

Draw AB and take C the middle point of AB.

Draw PC.

Then because in the triangles PCA, PCB,

PA is equal to PB, PC to PC, and AC to BC,
the two triangles are equal in all respects: (Prop. 8.)
therefore the angle PCA is equal to the angle PCB,
and since they are adjacent angles each is a right angle.

It follows that, if P be equidistant from A and B, it must lie on the straight line CP which bisects AB at right angles. Every point on CP satisfies this condition.

We may state the result of this proposition thus: *The locus of a point equidistant from two given points is the straight line which bisects at right angles the straight line joining the given points.*

EXERCISES.

1. The diagonals of a rhombus bisect each other at right angles.

2. Find in a given straight line a point equidistant from two given points. Is a solution always possible?

3. Find a point equidistant from three given points.

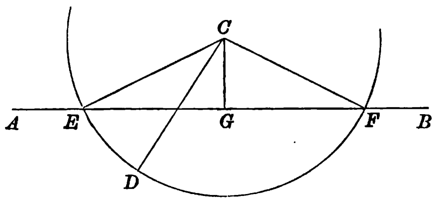
4. In the base BC of a triangle ABC any point D is taken. Draw a straight line such that, if the triangle ABC be folded along this straight line, the point A shall fall upon the point D.

PROPOSITION 12.

To draw a straight line at right angles to a given straight line from a given point without it.

Let *AB* be the given straight line, and *C* the given point without it:
it is required to draw from *C* a straight line at right angles to *AB*.

Construction. Take any point *D* on the side of *AB* away from *C*, draw *CD*,
and with *C* as centre and *CD* as radius, describe the circle *DEF*, (Post. 6.)
meeting *AB* (produced if necessary) at *E* and *F*.
Draw *CE*, *CF*,
and bisect the angle *ECF* by the straight line *CG* meeting *AB* at *G*: (Prop. 9.)
then *CG* is a straight line drawn as required.



Proof. Because in the triangles *ECG*, *FCG*,
EC is equal to *FC*,
and *CG* to *CG*,
and the angle *ECG* is equal to the angle *FCG*,
the triangles are equal in all respects; (Prop. 4.)
therefore the angle *CGE* is equal to the angle *CGF*,
and they are adjacent angles.
Therefore the straight lines *CG*, *AB* are at right angles to each other. (Def. 11.)

Wherefore, *CG has been drawn at right angles to the given straight line AB from the given point C without it.*

In the construction for this proposition it is said that the point D is to be taken on the side of AB *away from* C. This restriction is introduced as a means of ensuring the intersection of the circle DEF with the straight line AB.

EXERCISES.

1. Through two given points on opposite sides of a given straight line draw two straight lines, which shall meet in the given line and include an angle bisected by that line. In what case can there be more than one solution?

2. From two given points on the same side of a given straight line, draw two straight lines, which shall meet at a point in the given line and make equal angles with it.

3. Prove by the method of superposition that, if the perpendiculars on a given straight line from two points on the same side of it be equal, the straight line joining the points is parallel to the given line.

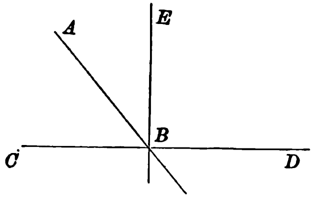
PROPOSITION 13.

The sum of the angles, which one straight line makes with another straight line on one side of it, is equal to two right angles.

Let the straight line *AB* make with the straight line *CD*, on one side of it, the angles *ABC*, *ABD*:
it is required to prove that the sum of these angles is equal to two right angles.

If the angle *ABC* be equal to the angle *ABD*,
each of them is a right angle,
and their sum is equal to two right angles.

CONSTRUCTION. If the angles *ABC*, *ABD* be not equal, from the point *B* draw *BE* at right angles to *CD*; (Prop. 11.)
BE cannot coincide with *BA*;
let it lie within the angle *ABD*.



PROOF. Now the angle *CBE* is the sum of the angles *CBA*, *ABE*;
to each of these equals add the angle *EBD*;
then the sum of the angles *CBE*, *EBD* is equal to the sum of the angles *CBA*, *ABE*, *EBD*.

Again, the angle *DBA* is equal to the sum of the angles *DBE*, *EBA*;
to each of these equals add the angle *ABC*;
then the sum of the angles *DBA*, *ABC* is equal to the sum of the angles *DBE*, *EBA*, *ABC*.

And the sum of the angles *CBE*, *EBD* has been proved to be equal to the sum of the same three angles.

Therefore the sum of the angles *CBE*, *EBD* is equal to the sum of the angles *DBA*, *ABC*.

But *CBE*, *EBD* are two right angles; (Constr.)

therefore the sum of the angles *DBA*, *ABC* is equal to two right angles.

Wherefore, *the sum of the angles* &c.

COROLLARY. *The sum of the four angles, which two intersecting straight lines make with one another, is equal to four right angles.*

EXERCISES.

1. Prove in the manner of Proposition 13 that, if *A*, *B*, *C*, *D* be four points in order on a straight line, the sum of *AB*, *BD* is equal to the sum of *AC*, *CD*.

2. If one of the four angles, which two intersecting straight lines make with one another, be a right angle, all the others are right angles.

3. Prove by the method of superposition that only one perpendicular can be drawn to a given straight line from a given point without it.

4. Prove by the method of superposition that, if two right-angled triangles have their hypotenuses equal and two other angles equal, the triangles are equal in all respects.

5. A given angle *BAC* is bisected; if *CA* be produced to *G* and the angle *BAG* bisected, the two bisecting lines are at right angles.

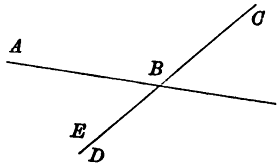
PROPOSITION 14.

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines are in the same straight line.

At the point *B* in the straight line *AB*, let the two straight lines *BC*, *BD*, on opposite sides of *AB*, make the adjacent angles *ABC*, *ABD* together equal to two right angles:

it is required to prove that *BD* is in the same straight line with *CB*.

Construction. Produce *CB* to *E*.



Proof. Because the straight line *AB* makes with the straight line *CBE*, on one side of it, the angles *ABC*, *ABE*, the sum of these angles is equal to two right angles. (Prop. 13.)

But the sum of the angles *ABC*, *ABD* is equal to two right angles.

Therefore the sum of the angles *ABC*, *ABD* is equal to the sum of the angles *ABC*, *ABE*.

From each of these equals take away the angle *ABC*;
then the angle *ABD* is equal to the angle *ABE*;
therefore the line *BD* coincides in direction with *BE*,
and is in the same straight line with *CB*.

Wherefore, *if at a point* &c.

DEFINITION. *Two angles, which are together equal to two right angles, are called* **supplementary angles**, *and each angle is said to be* **the supplement** *of the other.*

Two angles, which are together equal to one right angle, are called **complementary angles**, *and each angle is said to be* **the complement** *of the other.*

EXERCISES.

1. If E be the middle point of the diagonal AC of a quadrilateral $ABCD$, whose opposite sides are equal, B, E, D lie on a straight line.

2. If OA, OB, OC, OD be four straight lines drawn in order from O, such that the angles BOC, DOA are equal and also the angles AOB, COD, then the lines OA, OC are in the same straight line and also the lines OB, OD.

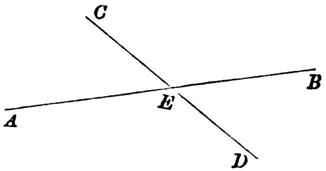
3. If it be possible within a quadrilateral $ABCD$, whose opposite sides are equal, to find a point E such that EA, EC are equal, and EB, ED are equal, then AEC, BED are straight lines.

4. If it be possible within a quadrilateral $ABCD$, whose opposite sides are equal, to find a point E, such that EA, EB, EC, ED are equal, then the quadrilateral is equiangular.

PROPOSITION 15.

If two straight lines intersect, vertically opposite angles are equal.

Let the two straight lines AB, CD intersect at E:
it is required to prove that the angle AEC is equal to the angle DEB, and the angle CEB to the angle AED.



PROOF. The sum of the angles CEA, AED, which AE makes with CD on one side of it, is equal to two right angles. (Prop. 13.)

Again, the sum of the angles AED, DEB, which DE makes with AB on one side of it, is equal to two right angles.

Therefore the sum of the angles CEA, AED is equal to the sum of the angles AED, DEB.

From each of these equals take away the common angle AED;

then the angle CEA is equal to the angle DEB.

Similarly it may be proved that the angle CEB is equal to the angle AED.

Wherefore, *if two straight lines* &c.

EXERCISES.

1. If the diagonals of a quadrilateral bisect one another, opposite sides are equal.

2. In a given straight line find a point such that the straight lines, joining it to each of two given points on the same side of the line, make equal angles with it.

3. A, B are two given points; CD, DE two given straight lines: find points P, Q in CD, DE, such that AP, PQ are equally inclined to CD, and PQ, QB equally inclined to DE.

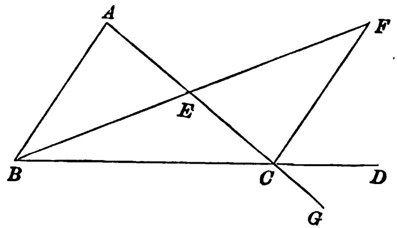
4. A straight line is drawn terminated by one of the sides of an isosceles triangle, and by the other side produced, and bisected by the base: prove that the straight lines thus intercepted between the vertex of the isosceles triangle, and this straight line, are together equal to the two equal sides of the triangle.

PROPOSITION 16.

An exterior angle of a triangle is greater than either of the interior opposite angles.

Let *ABC* be a triangle, and let *ACD* be the exterior angle made by producing the side *BC* to *D*:
it is required to prove that the angle *ACD* is greater than either of the interior opposite angles *CBA*, *BAC*.

CONSTRUCTION. Bisect *AC* at *E*. (Prop. 10.)
Draw *BE* and produce it to *F*,
making *EF* equal to *EB*, (Prop. 3.)
and draw *FC*.



PROOF. Because in the triangles *AEB*, *CEF*,
AE is equal to *CE*,
and *EB* to *EF*,
and the angle *AEB* is equal to the angle *CEF*,
the triangles are equal in all respects; (Prop. 4.)
therefore the angle *BAE* (or *BAC*) is equal to the angle *FCE*.

Now the angle *ECD* (or *ACD*) is greater than the angle *ECF*.

Therefore the angle *ACD* is greater than the angle *BAC*.

Similarly it can be proved that the angle *BCG*, which is made by producing *AC* and is equal to the angle *ACD*, is greater than the angle *ABC*.

Wherefore, *an exterior angle* &c.

EXERCISES.

1. Only one perpendicular can be drawn to a given straight line from a given point without it.

2. Shew by joining the angular point *A* of a triangle to any point in the opposite side *BC* between *B* and *C* that the angles *ABC*, *BCA* are together less than two right angles.

3. Not more than two equal straight lines can be drawn from a given point to a given straight line.

4. Prove by the method of superposition that, if a quadrilateral be equiangular, its opposite sides are equal.

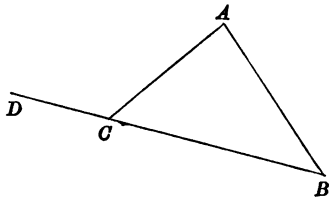
5. Prove by the method of superposition that two right-angled triangles, which have their hypotenuses equal and one side equal to one side, are equal in all respects.

PROPOSITION 17.

The sum of any two angles of a triangle is less than two right angles.

Let *ABC* be a triangle:
it is required to prove that the sum of any two of its angles is less than two right angles.

CONSTRUCTION. Produce any side *BC* to *D*.



PROOF. Because *ACD* is an exterior angle of the triangle *ABC*,
it is greater than the interior opposite angle *ABC*. (Prop. 16.)

To each of these unequals add the angle *ACB*:
then the sum of the angles *ACD*, *ACB* is greater than the sum of the angles *ABC*, *ACB*.

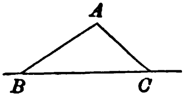
But the sum of the angles *ACD*, *ACB* is equal to two right angles. (Prop. 13.)

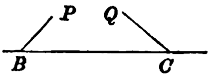
Therefore the sum of the angles *ABC*, *ACB* is less than two right angles.

Similarly it can be proved that the sum of the angles *BAC*, *ACB* is less than two right angles;
and also the sum of the angles *CAB*, *ABC*.

Wherefore, *the sum of any two angles* &c.

The theorem established in this proposition may be stated thus: If from two points B, C in the straight line BC two straight lines be drawn which meet at any point A, then the sum of the angles ABC, ACB is less than two right angles.





We shall assume as a postulate the converse of this theorem, which may be stated thus

If from two points B, C in the straight line BC two straight lines BP, CQ be drawn making the sum of the angles PBC, QCB on the same side of BC less than two right angles, the two lines BP, CQ will meet if produced far enough.

It may be observed that the theorem established in Proposition 17 proves that the lines PB, QC cannot meet when produced beyond B and C: if therefore the postulate just stated be allowed, it follows that the lines BP, CQ must meet when produced beyond P and Q.

The postulate which we here assume may be stated in general terms as follows

POSTULATE 9. *If the sum of the two interior angles, which two straight lines make with a given straight line on the same side of it, be not equal to two right angles, the two straight lines are not parallel.*

EXERCISES.

1. A triangle must have at least two acute angles.

2. Assuming Postulate 9, prove that any two straight lines drawn at right angles to two given intersecting straight lines must intersect.

3. Prove that a straight line drawn at right angles to a given straight line must intersect all straight lines which are not at right angles to the given straight line.

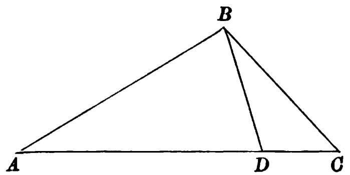
PROPOSITION 18.

When two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

Let ABC be a triangle, of which the side AC is greater than the side AB:
it is required to prove that the angle ABC is greater than the angle ACB.

CONSTRUCTION. From AC the greater of the two sides cut off AD equal to AB the less. (Prop. 3.)

Draw BD.



PROOF. Because ADB is an exterior angle of the triangle BDC,
it is greater than the interior opposite angle DCB. (Prop. 16.)

And because AB is equal to AD,
the angle ADB is equal to the angle ABD. (Prop. 5.)

Therefore the angle ABD is greater than the angle ACB.

But the angle ABC is greater than the angle ABD;
therefore the angle ABC is greater than the angle ACB.

Wherefore, *when two sides* &c.

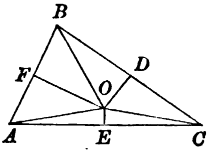
ADDITIONAL PROPOSITION.

The straight lines drawn at right angles to the sides of a triangle at their middle points meet in a point.

Let ABC be a triangle and D, E, F the middle points of the sides BC, CA, AB.

Draw EO, FO* at right angles to CA, AB.

Draw OA, OB, OC, OD.



Because in the triangles AEO, CEO,
AE is equal to CE, EO common, and the angle AEO is equal to the angle CEO,
the triangles are equal in all respects; (Prop. 4.)
therefore AO is equal to CO.
Similarly it can be proved that AO is equal to BO;
therefore BO is equal to CO.
Next because in the triangles BOD, COD,
BO is equal to CO and BD to CD and OD is common,
the triangles are equal in all respects; (Prop. 8.)
therefore the angle BDO is equal to the angle CDO,
and OD is at right angles to BC.

Wherefore the straight line drawn at right angles to BC at its middle point D passes through O, the intersection of the straight lines drawn at right angles to the other two sides at their middle points.

EXERCISES.

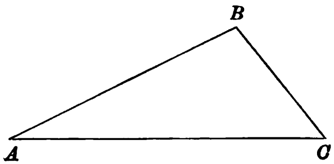
1. ABC is a triangle and the angle A is bisected by a straight line which meets BC at D; shew that BA is greater than BD, and CA greater than CD.
2. Prove that, if D be any point in the base BC between B and C of an isosceles triangle ABC, AD is less than AB.
3. Prove that, if AB, AC, AD be equal straight lines, and AC fall within the angle BAD, BD is greater than either BC or CD.
4. $ABCD$ is a quadrilateral of which AD is the longest side and BC the shortest; shew that the angle ABC is greater than the angle ADC, and that the angle BCD is greater than the angle BAD.
5. If the angle C of a triangle be equal to the sum of the angles A and B, the side AB is equal to twice the straight line joining C to the middle point of AB.

* We *assume* that the straight lines drawn at right angles to CA, AB at E and F meet. (See Exercise 2, page 51.)

PROPOSITION 19.

When two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

Let ABC be a triangle, of which the angle ABC is greater than the angle ACB:
it is required to prove that the side AC is greater than the side AB.



PROOF. AC must be either less than, equal to, or greater than AB.

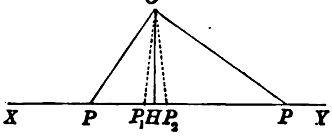
If AC were less than AB,
the angle ABC would be less than the angle ACB; (Prop. 18.)
but it is not;
therefore AC is not less than AB.

If AC were equal to AB,
the angle ABC would be equal to the angle ACB; (Prop. 5.)
but it is not;
therefore AC is not equal to AB.

Therefore AC must be greater than AB.

Wherefore, *when two angles* &c.

We leave it to the student to prove that, while a point P is moving along a straight line XY, the distance OP of the point P from a fixed point O outside the line is decreasing when P is moving towards H the foot of the perpendicular from O on the line, and that OP is increasing when P is moving away from H. Assuming the



truth of this proposition, it follows that OH is *less* than each of the two straight lines OP_1, OP_2 where P_1, P_2 are two positions of the point P close to H on either side of it. For this reason we say that OH is a **minimum** value of OP.

In the same way, if a geometrical quantity vary continuously, its magnitude in a position, where it is *greater* than in the positions close to it on either side, is called a **maximum** value.

It will be seen that, if a quantity vary continuously, there must be between any two *equal* values of the quantity at least one **maximum** or **minimum** value.

EXERCISES.

1. Prove that the hypotenuse of a right-angled triangle is greater than either of the other sides.

2. The base of a triangle is divided into two parts by the perpendicular from the opposite vertex: prove that each part of the base is less than the adjacent side of the triangle.

3. A straight line drawn from the vertex of an isosceles triangle to any point in the base produced is greater than either of the equal sides.

4. If D be any point in the side BC of a triangle ABC, then the greater of the sides AB, AC is greater than AD.

5. The perpendicular is the shortest straight line which can be drawn from a given point to a given straight line; and, of any two others, that which makes the smaller angle with the perpendicular is the shorter.

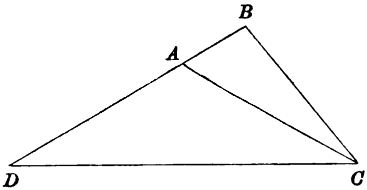
6. The base of a triangle whose sides are unequal is divided into two parts by the straight line bisecting the vertical angle: prove that the greater part is adjacent to the greater side.

PROPOSITION 20.

The sum of any two sides of a triangle is greater than the third side.

Let ABC be a triangle:
it is required to prove that the sum of any two sides of it is greater than the third side;
namely, the sum of CA, AB greater than BC;
the sum of AB, BC greater than CA;
the sum of BC, CA greater than AB.

CONSTRUCTION. Produce any side BA to D,
making AD equal to AC. (Prop. 3.)
Draw DC.



PROOF. Because AC is equal to AD,
the angle ADC is equal to the angle ACD. (Prop. 5.)
But the angle BCD is greater than the angle ACD.
Therefore the angle BCD is greater than the angle BDC.

And because in the triangle BCD,
the angle BCD is greater than the angle BDC;
BD is greater than BC. (Prop. 19.)

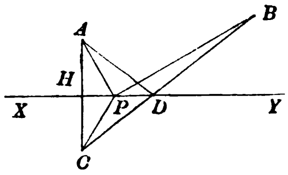
Now because DA is equal to AC,
BD, which is the sum of BA, AD, is equal to the sum of CA, AB.

Therefore the sum of CA, AB is greater than BC.

Similarly it can be proved that the sum of AB, BC is greater than CA; and that the sum of BC, CA is greater than AB.

Wherefore, *the sum of any two sides* &c.

The result of this proposition enables us to solve a great number of problems, of which the following is a specimen—*To find in a given straight line XY a point P such that the sum of its distances PA, PB from two given points A, B is a* **minimum.**



If the points A, B be on opposite sides of XY, the straight line AB intersects XY in the point required.

If A, B be on the same side of XY, draw AH perpendicular to XY; produce AH to C, so that HC is equal to HA.

Take any point P in XY.

Draw BDC, DA, PA, PB, PC. Then it is easily proved that AP is equal to CP, and AD to CD.

Therefore the sum of AP, PB is equal to the sum of CP, PB, and this is a *minimum* when P coincides with D. (Prop. 20.)

Therefore D is the point required.

From the diagram it is seen that the angle BDY is equal to the angle CDX, which is equal to the angle ADX.

It appears therefore that when the sum of PA, PB is a *minimum*, the lines PA, PB make equal angles with XY.

EXERCISES.

1. Prove that any three sides of any quadrilateral are greater than the fourth side.

2. If D be any point within a triangle ABC, the sum of DA, DB, DC is greater than half the perimeter of the triangle.

3. The sum of the four sides of any quadrilateral is greater than the sum of its two diagonals.

4. In a convex quadrilateral the sum of the diagonals is greater than the sum of either pair of opposite sides.

5. D is the middle point of BC the base of an isosceles triangle ABC, and E any point in AC. Prove that the difference of BD, DE is less than the difference of AB, AE.

6. The two sides of a triangle are together greater than twice the straight line drawn from the vertex to the middle point of the base.

7. Find in a given straight line a point such that the difference of its distances from two fixed points is a *maximum*.

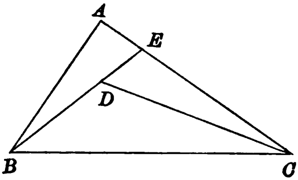
PROPOSITION 21.

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, the sum of these lines is less than the sum of the other two sides of the triangle, but they contain a greater angle.

Let ABC be a triangle; and from B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to a point D within the triangle:

it is required to prove that the sum of BD, DC is less than the sum of BA, AC, but the angle BDC is greater than the angle BAC.

CONSTRUCTION. Produce BD to meet AC at E.



PROOF. The sum of the two sides BA, AE of the triangle BAE is greater than the third side BE. (Prop. 20.)

To each of these unequals add EC;

then the sum of BA, AC is greater than the sum of BE, EC.

Again, the sum of the two sides CE, ED of the triangle CED is greater than the third side CD.

To each of these unequals add DB;

then the sum of CE, EB is greater than the sum of CD, DB.

And it has been proved that the sum of BA, AC is greater than the sum of BE, EC;

therefore the sum of BA, AC is greater than the sum of BD, DC.

Again, the exterior angle BDC of the triangle CDE is greater than the interior opposite angle CED. (Prop. 16.)

And the exterior angle CEB of the triangle ABE is greater than the interior opposite angle BAE;

therefore the angle BDC is greater than the angle BAC.

Wherefore, *if from the ends* &c.

EXERCISES.

1. If D be any point within a triangle ABC, the sum of DA, DB, DC is less than the perimeter of the triangle and greater than half the perimeter.

2. Prove that the perimeter of a triangle is less than the perimeter of any triangle which is drawn completely surrounding it.

3. If two triangles have a common base and equal vertical angles, the vertex of each triangle lies outside the other triangle.

4. If from the angles of a triangle ABC, straight lines AOD, BOE, COF be drawn through a point O within the triangle to meet the opposite sides, the perimeter of the triangle ABC is greater than two-thirds of the sum of AD, BE, CF.

5. ABD, ACD are two triangles on the same side of AD in which AC is greater than AB. Prove that, if the angles ABD, ACD be both right angles or be equal obtuse angles, then BD is greater than DC.

PROPOSITION 22.

To construct a triangle having its sides equal to three given straight lines.

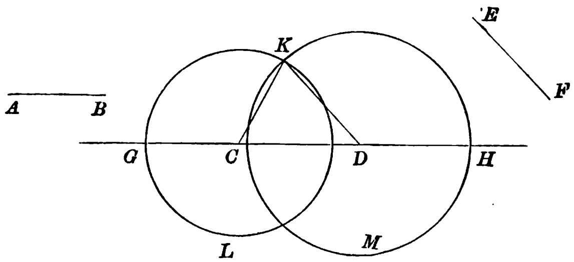
Let AB, CD, EF be the three given lines:
it is required to construct a triangle whose sides are equal to AB, CD, EF.

CONSTRUCTION. Produce one of the given lines CD both ways,

and cut off CG equal to AB, (Prop. 3.)
and DH to EF.

With C as centre and CG as radius describe the circle GKL, and with D as centre and DH as radius describe the circle HKM.

Let these circles intersect in K:
Draw CK, DK:
then CKD is a triangle drawn as required.



PROOF. Because C is the centre of the circle GKL,
CK is equal to CG;
and CG is equal to AB. (Constr.)
Therefore CK is equal to AB.

Again, because D is the centre of the circle HKM,
DK is equal to DH;
and DH is equal to EF. (Constr.)
Therefore DK is equal to EF.

Therefore the three lines KC, CD, DK are equal to the three AB, CD, EF respectively.

Wherefore, *the triangle KCD has been constructed having its sides equal to the three given straight lines AB, CD, EF.*

It may be observed that it is not possible to construct a triangle which shall have its sides equal to *any* three given straight lines. In Proposition 20 it has been proved that any two sides of a triangle are together greater than the third side. It follows therefore that it is impossible to construct a triangle having its sides equal to three given straight lines, except when the given straight lines are such that *any two of them are greater than the third* or *the greatest line is less than the sum of the other two.*

We see therefore that in this proposition we have to solve a problem, which admits of solution only when the *given* lines satisfy a certain condition.

We shall meet with many other problems in which the geometrical quantities *given* in the problem (for that reason generally called the data), must satisfy some condition in order that the problem may admit of solution. It will be a useful exercise for the student to investigate such conditions when they exist.

EXERCISES.

1. Prove that the two circles drawn in the construction of Proposition 22 will always intersect, provided that the sum of any two of the given straight lines is greater than the third.

2. How many different shaped triangles could be made of 8 different lines whose lengths are respectively 2, 2, 2, 3, 3, 4, 4, 5 inches?

3. Construct a right-angled triangle, having given the hypotenuse and one side.

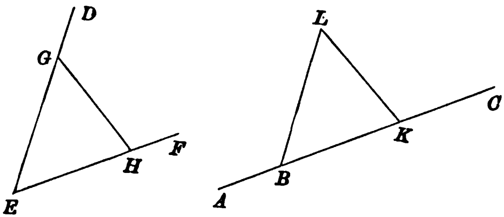
4. Construct a quadrilateral equal in all respects to a given quadrilateral.

PROPOSITION 23.

From a given point in a given straight line to draw a straight line making with the given straight line an angle equal to a given angle.

Let ABC be the given straight line, B the given point in it, and DEF the given angle:
it is required to draw from B a straight line making with ABC an angle equal to the angle DEF.

CONSTRUCTION. In ED, EF take any points G, H, and draw GH.
From BC cut off BK equal to EH, (Prop. 3.)
and construct the triangle LBK,
having the side BK equal to EH,
BL equal to EG,
and KL equal to HG: (Prop. 22.)
then BL is a straight line drawn as required.



PROOF. Because in the triangles BLK, EGH,
KB is equal to HE,
BL to EG,
and LK to GH,
the triangles are equal in all respects; (Prop. 8.)
therefore the angle KBL (or CBL) is equal to the angle HEG (or FED).

Wherefore, *from the given point B in the given straight line ABC a straight line BL has been drawn making with the straight line ABC an angle KBL equal to the given angle FED.*

EXERCISES.

1. Construct a triangle, having given the base and each of the angles at the base.

2. Make an angle double of a given angle.

3. If one angle of a triangle be equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

4. Construct a triangle, having given the base, one of the angles at the base, and the sum of the sides.

PROPOSITION 24.

If two sides of one triangle be equal to two sides of another and the angle contained by the two sides of the one be greater than the angle contained by the two sides of the other, the third side of the one is greater than the third side of the other.

Let ABC, DEF be two triangles, in which AB is equal to DE and AC to DF, and the angle BAC is greater than the angle EDF:

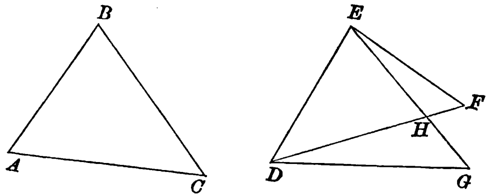
it is required to prove that the third side BC is greater than the third side EF.

Construction. Of the two sides DE, DF let DF be one which is not less than the other. From the point D in the straight line DE, draw DG making with DE

the angle EDG equal to the angle BAC, (Prop. 23.)

and make DG equal to DF. (Prop. 3.)

Draw EG meeting DF in H.



Proof. Because DF is not less than DE,

and DG is equal to DF,

DG is not less than DE.

And because in the triangle DEG,

DG is not less than DE,

the angle DEG is not less than the angle DGE.

(Props. 5 and 18.)

Next, because DHG is the exterior angle of the triangle DEH, it is greater than the interior opposite angle DEG.

(Prop. 16.)

Therefore the angle DHG is greater than the angle DGH.

And because in the triangle DHG,

the angle DHG is greater than the angle DGH,

DG is greater than DH. (Prop. 19.)

But DG is equal to DF.
Therefore DF is greater than DH,
or the point F lies outside the triangle DEG.

Next because the sum of DH, HG, two sides of the triangle DHG, is greater than the third side DG,
and the sum of FH, HE, two sides of the triangle EHF, is greater than the third side EF;
the sum of DH, HG, FH, HE is greater than the sum of DG, EF;
i.e. the sum of DF, EG is greater than the sum of DG, EF.

Take away the equals DF, DG;
then EG is greater than EF.

Now the triangles EDG, BAC are equal in all respects. (Prop. 4.)

Therefore BC, which is equal to EG, is greater than EF.

Wherefore, *if two sides* &c.

EXERCISES.

A point P moves along the circumference of a circle from one extremity A of a diameter AB to the other extremity B; prove that throughout the motion

(*a*) AP is increasing and BP is decreasing;

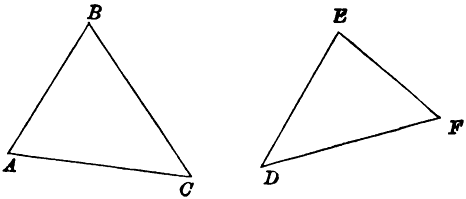
(*b*) if O be any point in AB nearer A than B, OP is increasing;

(*c*) if O be any point in BA produced, OP is increasing.

PROPOSITION 25.

If two sides of one triangle be equal to two sides of another, and the third side of the one be greater than the third side of the other, the angle opposite to the third side of the one is greater than the angle opposite to the third side of the other.

Let *ABC*, *DEF* be two triangles, in which *AB* is equal to *DE*, and *AC* to *DF*, and *BC* is greater than *EF*:
it is required to prove that the angle *BAC* is greater than the angle *EDF*.



PROOF. The angle *BAC* must be either greater than, equal to, or less than the angle *EDF*.

If the angle *BAC* were equal to the angle *EDF*,
BC would be equal to *EF*; (Prop. 4.)
but it is not;
therefore the angle *BAC* is not equal to the angle *EDF*.

Again, if the angle *BAC* were less than the angle *EDF*,
BC would be less than *EF*; (Prop. 24.)
but it is not;
therefore the angle *BAC* is not less than the angle *EDF*.

Therefore the angle *BAC* is greater than the angle *EDF*.

Wherefore, *if two sides* &c.

EXERCISES.

1. If D be the middle point of the side BC of a triangle ABC, in which AC is greater than AB, the angle ADC is an obtuse angle.

2. If in the sides AB, AC of a triangle ABC, in which AC is greater than AB, points D, E be taken such that BD, CE are equal, CD is greater than BE.

3. If in the sides AB, AC produced of a triangle ABC, in which AC is greater than AB, points D, E be taken such that BD, CE are equal, BE is greater than CD.

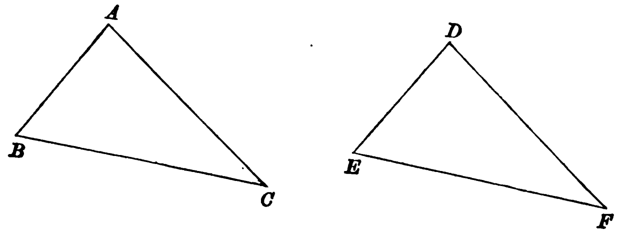
4. If in the side AB and the side AC produced of a triangle ABC points D and E be taken, such that BD, CE are equal, BE is greater than CD.

PROPOSITION 26. Part 1.

If two triangles have two angles of the one equal to two angles of the other, and the side adjacent to the angles in the one equal to the side adjacent to the angles in the other, the triangles are equal in all respects.

Let *ABC*, *DEF* be two triangles, in which the angle *ABC* is equal to the angle *DEF*, and the angle *BCA* is equal to the angle *EFD*, and the side *BC* adjacent to the angles *ABC*, *BCA* is equal to the side *EF* adjacent to the angles *DEF*, *EFD*:

it is required to prove that the triangles *ABC*, *DEF* are equal in all respects.



Proof. Because the sides *BC*, *EF* are equal, it is possible to shift the triangle *ABC*,

so that *BC* coincides with *EF*, *B* with *E* and *C* with *F*, (Test of Equality, page 5.)

and the triangles are on the same side of *EF*.

If this be done,

because *BC* coincides with *EF*,

and the angle *ABC* is equal to the angle *DEF*,

BA must coincide in direction with *ED*.

Similarly it may be proved that *CA* must coincide in direction with *FD*.

Therefore the point *A*, which is the intersection of *BA*, *CA*, must coincide with *D*, which is the intersection of *ED*, *FD*.

Next, because *A* coincides with *D*, and *B* with *E*,

AB must coincide with *DE*. (Post. 2.)

Similarly *AC* must coincide with *DF*.

Therefore the triangle ABC coincides with the triangle DEF, and is equal to it in all respects.

Wherefore, *if two triangles* &c.

EXERCISES.

1. If AD be the bisector of the angle BAC, and BDC be drawn at right angles to AD, AB is equal to AC.

2. AB, AC are any two straight lines meeting at A: through any point P draw a straight line meeting them at E and F, such that AE may be equal to AF.

3. If upon the same base AB two triangles BAC, ABD be constructed, having the angle BAC equal to ABD, and ABC equal to BAD, then the triangles BDC, ACD are equal in all respects.

4. If the opposite sides of a quadrilateral be equal, the diagonals bisect each other.

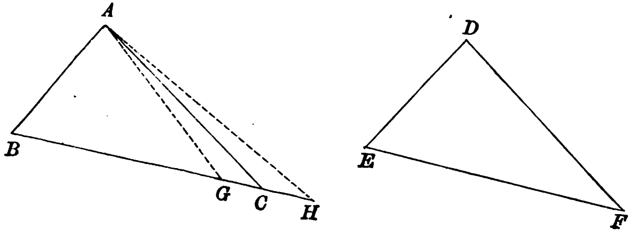
5. If the straight line bisecting the vertical angle of a triangle be at right angles to the base, the triangle is isosceles.

PROPOSITION 26. PART 2.

If two triangles have two angles of the one equal to two angles of the other, and the sides opposite to a pair of equal angles equal, the triangles are equal in all respects.

Let *ABC*, *DEF* be two triangles, in which the angle *ABC* is equal to the angle *DEF*, and the angle *BCA* equal to the angle *EFD*, and *BA* the side opposite to the angle *BCA* is equal to *ED* the side opposite to the angle *EFD*:

it is required to prove that the triangles *ABC*, *DEF* are equal in all respects.



PROOF. Because the sides *AB*, *DE* are equal, it is possible to shift the triangle *DEF*,

so that *DE* coincides with *AB*, *D* with *A* and *E* with *B*,

and the triangles are on the same side of *AB*.

If this be done,

because *ED* coincides with *BA*,

and the angle *DEF* is equal to the angle *ABC*,

EF must coincide in direction with *BC*.

Now *F* cannot coincide with any point *G* in *BC*,

since the angle *AGB* the exterior angle of the triangle *AGC* is greater than the interior and opposite angle *ACB*, (Prop. 16.)

which is equal to the angle *DFE*.

Again *F* cannot coincide with any point *H* in *BC* produced, since the interior and opposite angle *AHB* of the triangle *ACH* is less than the exterior angle *ACB*, (Prop. 16.)

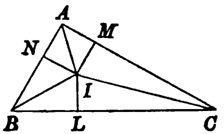
which is equal to the angle *DFE*.

Therefore F must coincide with C,
EF with BC, and DF with AC;
therefore the triangle DEF coincides with the triangle ABC,
and is equal to it in all respects.
Wherefore, *if two triangles* &c.

ADDITIONAL PROPOSITION.

The straight lines, which bisect the angles of a triangle, meet in a point.

Let ABC be a triangle.
Bisect the angles ABC, BCA by the straight lines BI, CI *.
Draw IL, IM, IN perpendicular to the sides.



Because in the triangles IBN, IBL
the angle IBN is equal to the angle IBL,
and the angle INB to the angle ILB,
and BI is common,
the triangles are equal in all respects: (Prop. 26, Part 2.)
therefore IN is equal to IL.
Similarly it can be proved that IM is equal to IL:
therefore IN is equal to IM.
Next because in the right-angled triangles IAN, IAM
the hypotenuse IA is common,
and IN is equal to IM,
the triangles are equal in all respects: (Exercise 5, page 49.)
therefore the angle IAN is equal to the angle IAM,
and IA is the bisector of the angle BAC.
Therefore the bisector of the angle BAC passes through the intersection of the bisectors of the angles ABC, BCA.

EXERCISES.

1. The perpendiculars let fall on two sides of a triangle from any point in the straight line bisecting the angle between them are equal to each other.

2. In a given straight line find a point such that the perpendiculars drawn from it to two given straight lines which intersect are equal.

3. Through a given point draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to each other.

* It is *assumed* that these lines intersect.

PROPOSITION 26 A.

If two triangles have two sides equal to two sides, and the angles opposite to one pair of equal sides equal, the angles opposite to the other pair are either equal or supplementary.

Let ABC, DEF be two triangles, in which AB is equal to DE, and BC to EF, and the angle BAC is equal to the angle EDF:

it is required to prove that the angles ACB, DFE are either equal or supplementary. (Def. page 45.)

Of the two sides AC, DF, let AC be not greater than DF.

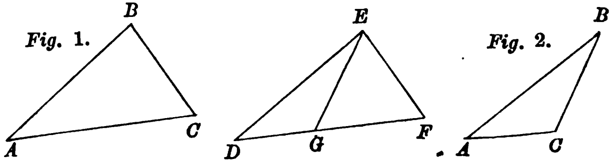


Fig. 1. *Fig.* 2.

Proof. Because the sides AB, DE are equal, it is possible to shift the triangle ABC,

so that AB coincides with DE,

A with D and B with E, (Test of Equality, page 5.)

and so that the two triangles ABC, DEF are on the same side of DE.

If this be done,

because AB coincides with DE,

and the angle BAC is equal to the angle EDF,

AC must coincide in direction with DF.

Because AC is not greater than DF,

C must coincide either (1) with F or (2) with G some point in DF.

(Fig 1.) If C coincide with F,

then BC coincides with EF, (Post. 2.)

and the triangle ABC with the triangle DEF,

and the two triangles are equal in all respects;

therefore the angle ACB is equal to the angle DFE.

(Fig. 2.) Again, if C coincide with G,

because BC is equal to EG, and EF is equal to BC,

EG is equal to EF.

And because in the triangle *EFG*,
EG is equal to *EF*,
the angle *EFG* is equal to the angle *EGF*. (Prop. 5.)
Now the angles *DGE*, *EGF* are together equal to two right angles, i.e. are supplementary; (Prop. 13.)
therefore the angles *DGE*, *EFG* are supplementary;
and the angle *DGE* is equal to the angle *ACB*;
therefore the angles *ACB*, *DFE* are supplementary.

Wherefore, *if two triangles* &c.

COROLLARY. *When two triangles have two sides equal to two sides, and the angles opposite to one pair of equal sides equal to one another, they are equal in all respects, provided that of the angles opposite to the second pair of equal sides,*

(1) *each be less than a right angle,*
(2) *each be greater than a right angle,*
or (3) *one of them be a right angle.*

EXERCISES.

1. If the straight line bisecting the vertical angle of a triangle also bisect the base, the triangle is isosceles.

2. If two given straight lines intersect, and a point be taken equally distant from each of them, it lies on one or other of the two straight lines which bisect the angles between the given straight lines.

3. Prove that two right-angled triangles are equal in all respects, if the hypotenuse and a side of the one be respectively equal to the hypotenuse and a side of the other.

4. If two exterior angles of a triangle be bisected, and from the point of intersection of the bisecting lines a straight line be drawn to the third angle, it bisects that angle.

5. If two triangles have two sides equal to two sides, and the angles opposite to the greater sides equal, the triangles are equal in all respects.

6. Construct a triangle having given two sides and the angle opposite to one of them. Is this always possible?

On Equal Triangles.

It is in many cases convenient to denominate the sides BC, CA, AB of a triangle ABC by the small letters a, b, c respectively. Here a, b, c stand for the sides of the triangle opposite to the angles A, B, C respectively.

Using this notation we may sum up the results of Propositions 4, 8, 26 Part 1, 26 Part 2, and 26 A as follows:

Two triangles ABC, $A'B'C'$ are equal in all respects,

(I) if $a=a'$, $b=b'$, and $C=C'$, (Prop. 4.)

(II) if $a=a'$, $b=b'$, and $c=c'$, (Prop. 8.)

(III) if $A=A'$, $B=B'$, and $c=c'$, (Prop. 26, Part 1.)

(IV) if $A=A'$, $B=B'$, and $a=a'$, (Prop. 26, Part 2.)

(V) if $a=a'$, $b=b'$, and $A=A'$, and if in addition

(1) B and B' be each less than a right angle,

or (2) B and B' be each greater than a right angle,

or (3) either B or B' be a right angle. (Prop. 26 A.)

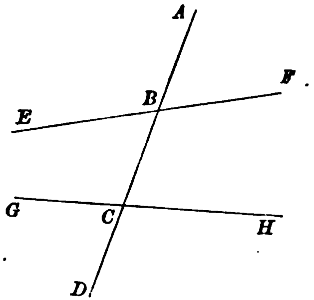
The six quantities, the angles A, B, C, and the sides a, b, c, are often denominated the **parts** of the triangle ABC.

It will be observed that the equality of three pairs of *parts* is always required to ensure the equality in all respects of two triangles, but that the equality of three pairs of parts is not always sufficient.

By the theorem of Proposition 32 it can be shewn that, if any two of the equations $A=A'$, $B=B'$, $C=C'$, be true, the third is also true: from this we conclude that the set of equations $A=A'$, $B=B'$, $C=C'$, is insufficient to determine the equality of the triangles, and that the two cases III. and IV. are virtually the same.

On the angles made by one straight line with two others.

When a straight line $ABCD$ intersects two other straight lines EBF, GCH,



the angles ABE, ABF, DCG, DCH *outside* the two lines EF, GH are called **exterior** angles;

the angles CBE, CBF, BCG, BCH *inside* the two lines EF, GH are called **interior** angles;

a pair of interior angles on opposite sides of $ABCD$ are called **alternate** angles.

There are two pairs of *alternate* angles in the diagram, EBC, BCH; CBF, BCG.

A pair of angles, one at B and the other at C, one exterior and the other interior, on the same side of $ABCD$ are sometimes called *corresponding* angles.

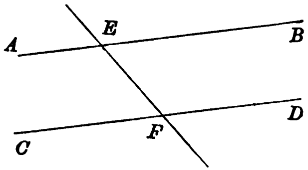
There are four pairs of corresponding angles, ABF, BCH; ABE, BCG; DCH, CBF; and DCG, CBE, the first angle in each pair being an exterior angle, and the second the interior.

PROPOSITION 27.

If a straight line, meeting two other straight lines in the same plane, make two alternate angles equal, the two straight lines are parallel.

Let the straight line *EF*, meeting the two straight lines *AB*, *CD* in the same plane, make the alternate angles *AEF*, *EFD* equal to one another:

it is required to prove that *AB*, *CD* are parallel.



PROOF. *AB*, *CD* cannot meet when produced beyond *B* and *D*; for if they did, the exterior angle *AEF* of the triangle formed by them and *EF* would be greater than the interior opposite angle *EFD* (Prop. 16.);

but it is not.

Similarly it can be proved that *AB*, *CD* cannot meet when produced beyond *A* and *C*.

But those straight lines in the same plane which do not meet however far they may be produced both ways, are parallel. (Def. 9.)

Therefore *AB*, *CD* are parallel.

Wherefore, *if a straight line* &c.

EXERCISES.

1. No two straight lines drawn from two angles of a triangle and terminated by the opposite sides can bisect one another.

2. Two straight lines at right angles to the same straight line are parallel.

3. Prove Proposition 27 by the method of superposition.

PROPOSITION 28.

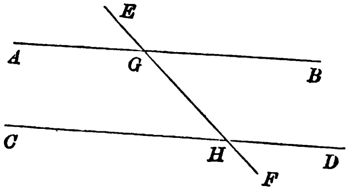
If a straight line intersecting two other straight lines, make an exterior angle equal to the interior and opposite angle on the same side of the line; or if it make two interior angles on the same side together equal to two right angles, the two straight lines are parallel.

Let the straight line *EF*, intersecting the two straight lines *AB*, *CD*,

(1) make the exterior angle *EGB* equal to the interior and opposite angle on the same side *GHD*,

or (2) make the interior angles on the same side *BGH*, *GHD* together equal to two right angles:

it is required to prove that *AB*, *CD* are parallel.



Proof. (1) Because the angle *EGB* is equal to the angle *GHD*,
and the angle *EGB* is equal to the angle *AGH*, (Prop. 15.)
the angle *AGH* is equal to the angle *GHD*;
and they are alternate angles;
therefore *AB*, *CD* are parallel. (Prop. 27.)

(2) Because the angles *BGH*, *GHD* are together equal to two right angles,
and the angles *AGH*, *BGH* are together equal to two right angles, (Prop. 13.)
the angles *AGH*, *BGH* are together equal to the angles *BGH*, *GHD*.
Take away the common angle *BGH*;
then the angle *AGH* is equal to the angle *GHD*;
and they are alternate angles;
therefore *AB*, *CD* are parallel. (Prop. 27.)

Wherefore, *if a straight line* &c.

EXERCISES.

1. If a straight line intersecting two other straight lines make two external angles on the same side of the line together equal to two right angles, the two straight lines are parallel.

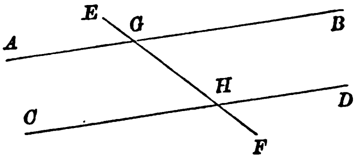
2. If a straight line intersecting two other straight lines make two external angles on opposite sides of the line equal, the two straight lines are parallel.

3. If a straight line intersecting two other straight lines make two corresponding angles equal, the two straight lines are parallel.

PROPOSITION 29.

If a straight line intersect two parallel straight lines, it makes alternate angles equal, it makes each exterior angle equal to the interior and opposite angle on the same side of the line, and it also makes interior angles on the same side together equal to two right angles.

Let the straight line *EF* intersect the two parallel straight lines *AB*, *CD*:
it is required to prove that
(1) the alternate angles *AGH*, *GHD* are equal,
(2) the exterior angle *EGB* is equal to the interior and opposite angle *GHD* on the same side of *EF*,
and (3) the two interior angles *BGH*, *GHD* on the same side of *EF* are together equal to two right angles.



Proof. (1) Because *AGH*, *BGH* are the angles which *EF* makes with *AB* on one side of it,
the sum of the angles *AGH*, *BGH* is equal to two right angles. (Prop. 13.)
Therefore, if the angles *AGH*, *GHD* were unequal,
the sum of the angles *BGH*, *GHD* would not be equal to two right angles;
and since these are the interior angles which the straight lines *AB*, *CD* make with *EF* on one side of it,
AB, *CD* would not be parallel. (Post. 9, page 51,)
But *AB*, *CD* are parallel;
therefore the angle *AGH* is equal to the angle *GHD*.
(2) But the angle *AGH* is equal to the angle *EGB*; (Prop. 15.)
therefore the angle *EGB* is equal to the angle *GHD*.
(3) Add to each of the equal angles *EGB*, *GHD* the angle *BGH*;

then the angles EGB, BGH are together equal to the angles BGH, GHD.

But the angles EGB, BGH are together equal to two right angles.

Therefore the angles BGH, GHD are together equal to two right angles.

Wherefore, *if a straight line &c.*

Corollary. *All the angles of a rectangle are right angles.* (See Def. 19.)

EXERCISES.

1. Any straight line parallel to the base of an isosceles triangle makes equal angles with the sides.

2. If through any point equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines, they will intercept equal portions of these parallel straight lines.

3. If the straight line bisecting an exterior angle of a triangle be parallel to a side, the triangle is isosceles.

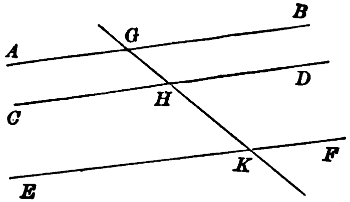
4. If DE, DF drawn from D any point in the base BC of an isosceles triangle ABC, to meet AB, AC in E, F be parallel to AC, AB, the perimeter of the parallelogram $AEDF$ is constant.

PROPOSITION 30.

Straight lines parallel to the same straight line are parallel to each other.

Let each of the straight lines AB, CD be parallel to EF: it is required to prove that AB, CD are parallel to one another.

CONSTRUCTION. Draw a straight line GHK intersecting AB, CD, EF in G, H, K respectively.



PROOF. Because GHK intersects the parallels AB, EF, the angle GKF is equal to the angle AGH. (Prop. 29.)

Again, because GK intersects the parallels CD, EF, the angle GHD is equal to the angle GKF. (Prop. 29.)

Therefore the angle AGH is equal to the angle GHD; and they are alternate angles;

therefore AB is parallel to CD. (Prop. 27.)

Wherefore, *straight lines* &c.

EXERCISES.

1. Two intersecting straight lines cannot both be parallel to the same straight line.

2. Only one straight line can be drawn through a given point parallel to a given straight line.

3. If two straight lines, each of which is parallel to a third straight line, meet, the two lines are coincident throughout their length.

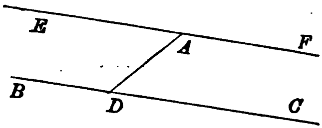
4. If a straight line intersect one of two parallel straight lines, it must intersect the other.

PROPOSITION 31.

To draw through a given point a straight line parallel to a given straight line.

Let A be the given point, and BC the given straight line:
it is required to draw through A a straight line parallel to BC.

CONSTRUCTION. In BC take any point D, and draw AD; from the point A in the straight line AD on the side of AD remote from C draw AE making the angle DAE equal to the angle ADC; (Prop. 23.)
and produce the straight line EA to F:
then EF is the straight line required.



PROOF. Because the straight line AD meets the two straight lines BC, EF,
and makes the alternate angles EAD, ADC equal,
EF is parallel to BC. (Prop. 27.)

Wherefore, *the straight line EAF has been drawn through the given point A, parallel to the given straight line BC.*

EXERCISES.

1. Find a point B in a given straight line CD, such that, if AB be drawn to B from a given point A, the angle ABC will be equal to a given angle.

2. Draw through a given point between two intersecting straight lines a straight line so that it is bisected at the point.

3. $ABCD$ is a quadrilateral having BC parallel to AD; shew that its area is the same as that of the parallelogram which can be formed by drawing through the middle point of DC a straight line parallel to AB.

4. AC, BC are two given straight lines: it is required to draw a straight line from a given point P to AC, so that it is bisected by BC.

5. Construct a triangle having given two angles, and the length of the perpendicular from the third angle on the opposite side.

6. Construct a right-angled triangle, having given one side and the angle opposite.

PROPOSITION 32.

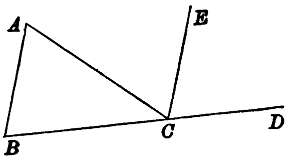
An exterior angle of a triangle is equal to the sum of the two interior opposite angles; and the sum of the three interior angles of a triangle is equal to two right angles.

Let ABC be a triangle:

it is required to prove that (1) the exterior angle ACD made by producing the side BC is equal to the sum of the two interior opposite angles CAB, ABC,

and (2) the sum of the three interior angles ABC, BCA, CAB is equal to two right angles.

CONSTRUCTION. Through the point C draw CE parallel to BA. (Prop. 31.)



PROOF. (1) Because AC meets the parallels BA, CE, the alternate angles BAC, ACE are equal. (Prop. 29.)

Again, because BD meets the parallels BA, CE,

the exterior angle ECD is equal to the interior opposite angle ABC. (Prop. 29.)

And the angle ACE was proved to be equal to the angle BAC;

therefore the whole angle ACD is equal to the sum of the two angles CAB, ABC.

(2) To each of these equals add the angle BCA;

then the sum of the angles ACD, ACB is equal to the sum of the three angles ABC, BCA, CAB.

But the sum of the angles ACD, ACB is equal to two right angles; (Prop. 13.)

therefore also the sum of the angles ABC, BCA, CAB is equal to two right angles.

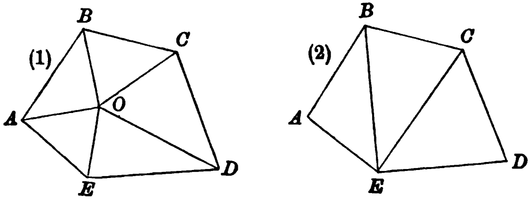
Wherefore, *An exterior angle* &c.

COROLLARY. *The sum of the interior angles of any convex rectilineal figure of* n *sides is less by four right angles than* 2n *right angles.*

This may be proved in either of the following ways:

In fig. (1), where straight lines are drawn from any point O within the figure to the vertices, the angles of the n triangles so formed are equal to the angles of the figure together with the angles at O, which are equal to four right angles.

In fig. (2), where all the diagonals from one vertex E are drawn, the angles of the $n-2$ triangles so formed are together equal to the angles of the figure.



EXERCISES.

1. Straight lines AD, BE, CF are drawn within the triangle ABC making the angles DAB, EBC, FCA all equal to one another. If AD, BE, CF do not meet in a point, the angles of the triangle formed by them are equal to those of the triangle ABC.

2. Trisect a right angle.

3. Trisect a quarter of a right angle.

4. If A be the vertex of an isosceles triangle ABC, and BA be produced to D, so that AD is equal to BA, and DC be drawn: then BCD is a right angle.

5. A straight line drawn at right angles to BC the base of an isosceles triangle ABC cuts AB in D and CA produced in E: prove that AED is an isosceles triangle.

6. Construct a right-angled triangle having given the hypotenuse and the sum of the sides.

7. The line joining the right angle of a right-angled triangle to the middle point of the hypotenuse is equal to half the hypotenuse.

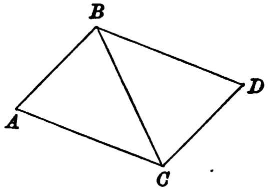
8. The locus of the vertices of all right-angled triangles which have a common hypotenuse is a circle.

PROPOSITION 33.

If two sides of a convex quadrilateral be equal and parallel, the other sides are equal and parallel.

Let *ABDC* be a quadrilateral, in which the sides *AB*, *CD* are equal and parallel:
it is required to prove that the sides *AC*, *BD* are equal and parallel.

CONSTRUCTION. Draw one of the diagonals *BC*.



PROOF. Because *AB* is parallel to *CD*,
and *BC* meets them,
the alternate angles *ABC*, *BCD* are equal. (Prop. 29.)
Because in the triangles *ABC*, *DCB*,
AB is equal to *DC*,
and *BC* to *CB*,
and the angle *ABC* to the angle *DCB*,
the triangles are equal in all respects; (Prop. 4.)
therefore the angle *ACB* is equal to the angle *DBC*,
and *CA* to *BD*.

And because the straight line *BC* meets the two straight lines *AC*, *BD*, and makes the alternate angles *ACB*, *CBD* equal to one another,
AC is parallel to *BD*. (Prop. 27.)
And it was proved to be equal to it.

Wherefore, *if two sides* &c.

EXERCISES.

1. Draw a straight line so that the part intercepted between two given straight lines is equal to one given straight line and parallel to another.

2. If a quadrilateral have two of its opposite sides parallel, and the two others equal but not parallel, any two of its opposite angles are together equal to two right angles.

3. If a straight line which joins the extremities of two equal straight lines, not parallel, make the angles on the same side of it equal to each other, the straight line which joins the other extremities will be parallel to the first.

4. If from D any point in the base BC of an isosceles triangle ABC, DE, DF be drawn perpendicular to the sides, then the sum of DE, DF is constant.

PROPOSITION 34.

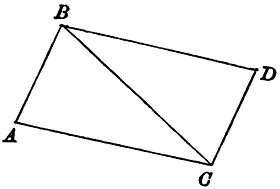
Opposite sides of a parallelogram are equal, and opposite angles are equal; and a diagonal of a parallelogram bisects its area.

Let *ACDB* be a parallelogram, of which *BC* is a diagonal:

it is required to prove that (1) opposite sides are equal, *AB* to *CD*, and *AC* to *BD*;

(2) opposite angles are equal, *BAC* to *BDC* and *ABD* to *ACD*;

and (3) the diagonal *BC* bisects the area of the parallelogram.



Proof. Because *AB* is parallel to *CD*, and *BC* meets them,

the alternate angles *ABC*, *BCD* are equal. (Prop. 29.)

And because *AC* is parallel to *BD*, and *BC* meets them,

the alternate angles *ACB*, *CBD* are equal. (Prop. 29.)

Now because in the two triangles *ABC*, *DCB*,

the angle *ABC* is equal to the angle *DCB*,

and the angle *BCA* to the angle *CBD*,

and the side *BC* adjacent to the equal angles in each is common to both,

the triangles are equal in all respects. (Prop. 26, Part 1.)

Therefore *AB* is equal to *DC*, *AC* equal to *DB*,

and the angle *BAC* equal to the angle *CDB*.

And because the angle *ABC* is equal to the angle *DCB*,

and the angle *CBD* to the angle *BCA*,

the whole angle *ABD* is equal to the whole angle *DCA*.

And the angle *BAC* has been proved to be equal to the angle *CDB*.

Therefore in the parallelogram *AD* (1) opposite sides are equal and (2) opposite angles are equal.

Again, it has been proved
that the triangles ABC, DCB are equal in all respects:
therefore (3) the diagonal BC bisects the area of the parallelogram AD.

Wherefore, *opposite sides* &c.

COROLLARY 1. *All the sides of a square are equal.*

COROLLARY 2. *The angles made by a pair of straight lines are equal to the angles made by any pair of straight lines parallel to them.*

A parallelogram $ABCD$ is often spoken of as the parallelogram AC, or the parallelogram BD, or more simply as AC or BD, when there is no danger of confusion with the diagonal AC or with the diagonal BD.

EXERCISES.

1. Prove that, if the diagonals of a quadrilateral bisect one another, the quadrilateral is a parallelogram. Prove also the converse.

2. If two sides of a quadrilateral be parallel and the other two equal but not parallel, the diagonals are equal.

3. If in a quadrilateral the diagonals be equal and two sides be parallel, the other sides are equal.

4. Find in a side of a triangle the point from which straight lines drawn parallel to the other sides of the triangle and terminated by them are equal.

5. Prove that every straight line which bisects the area of a parallelogram must pass through the intersection of its diagonals.

6. Construct a triangle whose angles shall be equal to those of a given triangle, and whose area shall be four times the area of the given triangle.

7. $ABCD$ is a parallelogram having the side AD double of AB: the side AB is produced both ways to E and F till each produced part equals AB, and straight lines are drawn from C and D to E and F so as to cross within the figure: shew that they will meet at right angles.

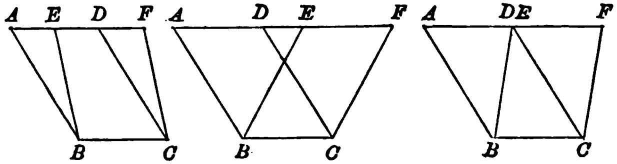
8. If O be any point within a parallelogram $ABCD$, the sum of the triangles OAB, OCD is half the parallelogram.

9. Divide a given straight line into n equal parts, where n is a whole number.

PROPOSITION 35.

Two parallelograms, which have one side common and the sides opposite to the common side in a straight line, are equal in area.

Let *ABCD*, *EBCF* be two parallelograms, which have a common side *BC*, and the sides *AD*, *EF* in a straight line:
it is required to prove that *ABCD*, *EBCF* are equal in area.



PROOF. Because *ABCD* is a parallelogram,
AB is equal to *DC*, (Prop. 34.)
and because *EBCF* is a parallelogram,
BE is equal to *CF*;
and because *AB* is parallel to *DC*,
and *BE* to *CF*,
the angle *ABE* is equal to the angle *DCF*.
(Prop. 34, Coroll. 2.)
And because in the triangles *ABE*, *DCF*,
AB is equal to *DC*,
and *BE* to *CF*,
and the angle *ABE* to the angle *DCF*,
the triangles are equal in all respects. (Prop. 4.)
Take from the area *ABCF*, the equal areas *FDC*, *EAB*;
then the remainders are equal,
that is, the parallelograms *ABCD*, *EBCF* are equal in area.

Wherefore, *two parallelograms* &c.

The propositions in the remaining part of the First Book of Euclid and those in the Second Book relate chiefly to cases of equality of the areas of two figures.

The test of equality to which we have hitherto always appealed has been that of the possibility of shifting one figure so that it exactly coincides with the other. In this case the figures are equal in all respects, but we say that two figures are *equal in area* also, when it is possible to shift all the parts of the area of one figure, so that they together exactly fit the area of the second figure.

It will be observed that this is the test made use of in Proposition 35.

For the future we shall often, when there is no danger of ambiguity, speak of the equality of two figures when we mean only equality of area, and we shall often speak of a figure when we mean only the area of the figure.

EXERCISES.

1. Construct a rectangle equal to a given parallelogram.

2. Construct a rhombus equal to a given parallelogram.

3. Construct a parallelogram to be equal to a given parallelogram in area and to have its sides equal to two given straight lines. Is this always possible?

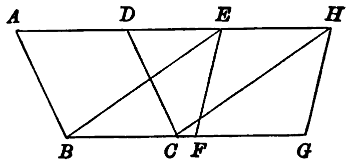
PROPOSITION 36.

Two parallelograms, which have two sides equal and in a straight line and also have the sides opposite to the equal sides in a straight line, are equal.

Let $ABCD$, $EFGH$ be two parallelograms, which have their sides BC, FG equal and in a straight line, and also their sides AD, EH in a straight line:

it is required to prove that $ABCD$, $EFGH$ are equal.

CONSTRUCTION. Draw BE, CH.



PROOF. Because BC is equal to FG,
and FG to EH, (Prop. 34.)
BC is equal to EH;
and they are parallel.

Because the two sides BC, EH of the convex quadrilateral $EBCH$ are equal and parallel,
the other sides BE, CH are equal and parallel; (Prop. 33.)
therefore $EBCH$ is a parallelogram.

Now because $EBCH$ and $ABCD$ have the side BC common, and the sides AD, EH in a straight line,
$EBCH$ is equal to $ABCD$. (Prop. 35.)

Similarly it can be proved that $EBCH$ is equal to $EFGH$.

Therefore the parallelograms $ABCD$, $EFGH$ are equal.

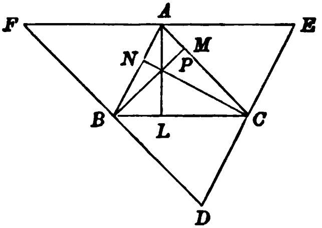
Wherefore, *two parallelograms* &c.

ADDITIONAL PROPOSITION.

The straight lines, drawn from the vertices of a triangle perpendicular to the opposite sides, meet in a point.*

Let ABC be a triangle, and AL, BM, CN be drawn perpendicular to BC, CA, AB respectively.

Draw the straight lines FAE, DBF, ECD parallel to BC, CA, AB respectively.



Because BE is a parallelogram,
AE is equal to BC; (Prop. 34.)
and because CF is a parallelogram,
FA is equal to BC;
therefore FA is equal to AE.
Again, because AL meets the parallels FAE, BLC,
the angle FAL is equal to the angle ALC. (Prop. 29.)
But the angle ALC is a right angle;
therefore the angle FAL is a right angle.

Therefore AL is the straight line drawn at right angles to FE at its middle point.

Similarly it can be proved that BM, CN are the straight lines drawn at right angles to FD, DE at their middle points.

Now AL, BM, CN the straight lines drawn at right angles to the sides of the triangle DEF at their middle points meet in a point. (Add. Prop., page 53.)

Therefore AL, BM, CN the straight lines drawn from the vertices of the triangle ABC perpendicular to the opposite sides meet in a point.

EXERCISES.

1. Construct a parallelogram to be equal to a given parallelogram and to have one of its sides in a given straight line.

2. Construct a parallelogram to be equal to a given parallelogram and to have two of its sides in two given straight lines.

* This point is often called **the orthocentre** of the triangle.

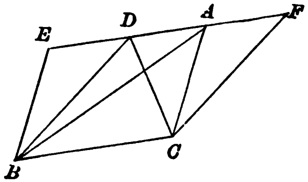
PROPOSITION 37.

Two triangles, which have one side common and the angular points opposite to the common side on a straight line parallel to it, are equal.

Let ABC, DBC be two triangles, which have a common side BC, and their angular points A, D on a straight line AD parallel to BC:

it is required to prove that the triangles ABC, DBC are equal.

CONSTRUCTION. Through B draw BE parallel to CA, and through C draw CF parallel to BD, (Prop. 31.) meeting AD (produced if necessary) in E and F.



PROOF. Because the parallelograms $EBCA$, $DBCF$ have a common side BC and the sides EA, DF in a straight line,

$EBCA$ is equal to $DBCF$. (Prop. 35.)

And because the diagonal AB bisects the parallelogram $EBCA$,

the triangle ABC is half of $EBCA$; (Prop. 34.)

and because the diagonal DC bisects the parallelogram $DBCF$,

the triangle DBC is half of $DBCF$.

Now the halves of equals are equal.

Therefore the triangles ABC, DBC are equal.

Wherefore, *two triangles* &c.

EXERCISES.

1. If P be a point within a parallelogram $ABCD$, the difference of the triangles PAB, PAD is equal to the triangle PAC.

2. If P be a point outside a parallelogram $ABCD$, the sum of the triangles PAB, PAD is equal to the triangle PAC.

3. AB and ECD are two parallel straight lines: BF, DF are drawn parallel to AD, AE respectively: prove that the triangles ABC, DEF are equal to one another.

4. ABC is a given triangle: construct a triangle of equal area, having AB for base and its vertex in a given straight line.

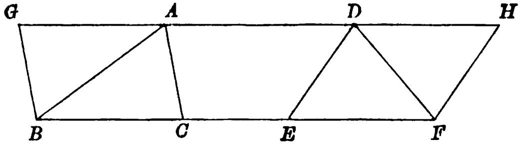
5. Points A, B, C are taken, one on each of three parallel straight lines: BC, CA, AB meet the lines through A, B, C respectively in a, b, c: prove that each of the triangles ABC, Abc, Bca, Cab, is equal to half the triangle abc.

PROPOSITION 38.

Two triangles, which have two sides equal and in a straight line and also have the angular points opposite to the equal sides on a straight line parallel to it, are equal.

Let *ABC*, *DEF* be two triangles, which have their sides *BC*, *EF* equal and in a straight line, and their angular points *A*, *D*, on a straight line *AD* parallel to *BF*: it is required to prove that the triangles *ABC*, *DEF* are equal.

CONSTRUCTION. Through *B* draw *BG* parallel to *CA*, and through *F* draw *FH* parallel to *ED*, meeting *AD* (produced if necessary) in *G* and *H*.



PROOF. Because the parallelograms *GBCA*, *DEFH* have their sides *BC*, *EF* equal and in a straight line, and also their sides *GA*, *DH* in a straight line, they are equal to one another. (Prop. 36.)

Because the diagonal *AB* bisects the parallelogram *GBCA*, the triangle *ABC* is half of *GBCA*; (Prop. 34.) and because the diagonal *DF* bisects the parallelogram *DEFH*, the triangle *DEF* is half of *DEFH*.

Now the halves of equals are equal;

therefore the triangles *ABC*, *DEF* are equal.

Wherefore, *two triangles* &c.

COROLLARY. *Two triangles, which have two sides equal and in a straight line and also have the angular points opposite to the equal sides coincident, are equal.*

EXERCISES.

1. $ABCD$ is a parallelogram; from any point P in the diagonal BD the straight lines PA, PC are drawn. Shew that the triangles PAB and PCB are equal in area.

2. The three sides of a triangle are bisected, and the points of bisection are joined; prove that the triangle is divided into four triangles, which are all equal to one another.

3. If the sides BC, CA, AB of a triangle ABC be produced to A', B', C' respectively, so that $CA'=BC$, $AB'=CA$, $AB=BC'$, prove that the area of the triangle $A'B'C'$ is seven times that of the triangle ABC.

4. Make a triangle such as to be equal to a given parallelogram, and to have one of its angles equal to a given angle.

5. If the sides AB, BC, CA of a triangle ABC be respectively bisected in c, a, b, and Aa, Cc intersect in P: then BPb is a straight line.

6. The sides AB, AC of a triangle are bisected in D, E: CD, BE intersect in F. Prove that the triangle BFC is equal to the quadrilateral $ADFE$.

7. If AB, $PQRS$, CD be three parallel straight lines and P, Q, R, S be situate on AC, AD, BC, BD respectively, then PQ is equal to RS, and PR to QS.

8. A', B', C' are the middle points of the sides of the triangle ABC, and through A, B, C are drawn three parallel straight lines meeting $B'C'$, $C'A'$, $A'B'$ in a, b, c respectively; prove that the triangle abc is half the triangle ABC and that bc passes through A, ca through B, ab through C.

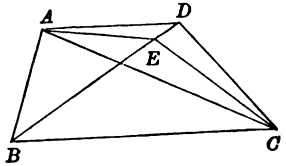
PROPOSITION 39.

If two equal triangles have a common side and lie on the same side of it, the angular points opposite to the common side lie on a straight line parallel to it.

Let *ABC*, *DBC* be two equal triangles, which have a common side *BC*, and lie on the same side of *BC*:

it is required to prove that the angular points *A*, *D* opposite to the side *BC* lie on a straight line parallel to *BC*.

CONSTRUCTION. Draw *AD*, and in *BD* or *BD* produced take any point *E* other than *D*, and draw *AE*, *EC*.



PROOF. Because the triangle *DBC* is not equal to the triangle *EBC*,

and the triangle *ABC* is equal to the triangle *DBC*,

the triangle *ABC* is not equal to the triangle *EBC*.

If *AE* were parallel to *BC*,

the triangle *ABC* would be equal to the triangle *EBC*; (Prop. 37.)

but they are not equal;

therefore *AE* is not parallel to *BC*.

But it is possible to draw a straight line through *A* parallel to *BC*; (Prop. 31.)

therefore *AD* is parallel to *BC*.

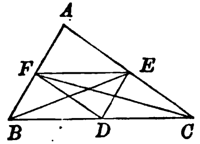
Wherefore, *if two equal triangles* &c.

Additional Proposition.

Each side of a triangle is double of the straight line joining the middle points of the other sides and is parallel to it.

Let ABC be a triangle, D, E, F the middle points of the sides BC, CA, AB.

Draw BE, CF, EF, FD, DE.



Because the two triangles BFC, AFC have their sides BF, AF equal and in a straight line, and the point C common,

the triangles are equal; (Prop. 38, Coroll.)

therefore the triangle BFC is half of the triangle ABC.

Similarly it can be proved that the triangle BEC is half of the triangle ABC.

Therefore the triangle BFC is equal to the triangle BEC.

Next because the triangles BFC, BEC are equal and have a common side BC, the straight line FE joining their vertices is parallel to BC. (Prop. 39.)

Similarly it can be proved that DF is parallel to CA and ED to AB.

Again because $BFED$ is a parallelogram,

BD is equal to FE. (Prop. 34.)

And because $DFEC$ is a parallelogram,

DC is equal to FE:

therefore BC is double of FE.

EXERCISES.

1. The middle points of the sides of any quadrilateral are the angular points of a parallelogram.

2. Of equal triangles on the same base, the isosceles triangle has the least perimeter.

3. Two triangles of equal area stand on the same base and on opposite sides: shew that the straight line joining their vertices is bisected by the base or the base produced.

4. The triangle ABC is double of the triangle EBC: shew that, if AE, BC produced if necessary meet in D, then AE is equal to ED.

5. If the straight lines joining the middle points of two of the sides of a triangle to the opposite vertices be equal, the triangle is isosceles.

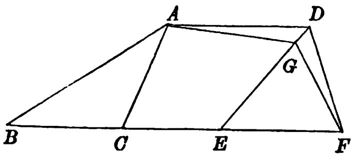
PROPOSITION 40.

If two equal triangles have two sides equal and in a straight line, and if the triangles lie on the same side of this line, the angular points opposite to the equal sides lie on a straight line parallel to the first straight line.

Let ABC, DEF be two equal triangles, which have equal sides BC, EF in a straight line and lie on the same side of BF:

it is required to prove that the angular points A, D opposite to BC, EF lie on a straight line parallel to BF.

CONSTRUCTION. Draw AD, and in ED or ED produced take any point G other than D, and draw AG, GF.



PROOF. Because the triangle DEF is not equal to the triangle GEF,

and the triangle ABC is equal to the triangle DEF,

the triangle ABC is not equal to the triangle GEF.

If AG were parallel to BF,

the triangle ABC would be equal to the triangle GEF; (Prop. 38.)

but they are not equal;

therefore AG is not parallel to BF.

But it is possible to draw a straight line through A parallel to BF; (Prop. 31.)

therefore AD is parallel to BF.

Wherefore, *if two equal triangles* &c.

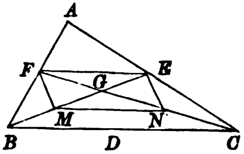
ADDITIONAL PROPOSITION.

The straight lines joining the vertices of a triangle to the middle points of the opposite sides meet in a point which is for each line the point of trisection further from the vertex.*

Let ABC be a triangle, and D, E, F be the middle points of the sides BC, CA, AB.

Draw BE, CF and let them intersect in G.

Bisect BG, CG in M, N, and draw FM, MN, NE.



In the triangle ABC,

BC is double of FE and is parallel to it. (Add. Prop., page 101.)

In the triangle GBC,

BC is double of MN and is parallel to it.

Therefore FE is equal and parallel to MN. (Prop. 30.)

Therefore $FMNE$ is a parallelogram. (Prop. 33.)

Now the diagonals of a parallelogram bisect each other. (Exercise 1, page 91.)

Therefore GE is equal to GM, which is equal to MB.

Therefore BG is double of GE.

Similarly CG is double of GF.

Similarly it can be proved that AD passes through G, and that AG is double of GD.

EXERCISES.

1. A point P is taken within a quadrilateral $ABCD$: prove that, if the sum of the areas of the triangles PAB, PCD be independent of the position of P, $ABCD$ is a parallelogram.

2. The locus of a point P such that the sum of the areas of the two triangles PAB, PBC is constant, is a straight line parallel to AC.

3. AB, CD are two given straight lines: the locus of a point P such that the sum of the two triangles PAB, PCD is constant, is a straight line.

4. Trisect a given straight line.

* This point is often called **the centre of gravity** or **the centroid** of the triangle.

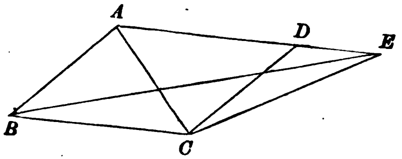
PROPOSITION 41.

If a parallelogram and a triangle have a common side, and the angular point of the triangle opposite to the common side lie on the same straight line as the opposite side of the parallelogram, the parallelogram is double of the triangle.

Let $ABCD$ be a parallelogram and EBC be a triangle which have a common side BC, and let the angular point E of the triangle lie in the same straight line as the side AD of the parallelogram :

it is required to prove that the parallelogram $ABCD$ is double of the triangle EBC.

CONSTRUCTION. Draw AC.



PROOF. Because the triangles ABC, EBC, have a common side BC, and AE is parallel to BC,

the triangles ABC, EBC are equal. (Prop. 37.)

And because the diagonal AC bisects the parallelogram $ABCD$,

the parallelogram $ABCD$ is double of the triangle ABC. (Prop. 34.)

Therefore the parallelogram $ABCD$ is double of the triangle EBC.

Wherefore, *if a parallelogram* &c.

EXERCISES.

1. *ABCD* is a parallelogram; from *D* draw any straight line *DFG* meeting *BC* at *F* and *AB* produced at *G*; draw *AF* and *CG*: shew that the triangles *ABF*, *CFG* are equal.

2. If *P* be a point in the side *AB*, and *Q* a point in the side *DC* of a parallelogram *ABCD*, then the triangles *PCD*, *QAB* are equal in area.

3. The area of any convex quadrilateral is double that of the parallelogram whose vertices are the middle points of the sides of the quadrilateral.

4. The sides *BC*, *CA*, *AB* of a triangle *ABC* are trisected in the points *D*, *d*; *E*, *e*; *F*, *f* respectively: prove that the area of the hexagon *DdEeFf* is two-thirds that of the triangle *ABC*.

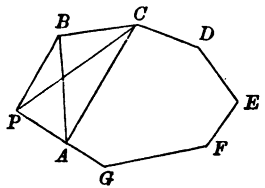
PROPOSITION 41 A.

To construct a triangle equal to a given rectilineal figure.

Let $ABCDEFG$ be the given rectilineal figure:
it is required to construct a triangle equal to $ABCDEFG$.

CONSTRUCTION. Draw one of the diagonals AC such that with two adjacent sides of the figure AB, BC it forms a triangle ABC.

Through the vertex B draw BP parallel to CA, to meet GA produced in P. Draw PC.



PROOF. Because the triangles PAC, BAC have a common side AC, and their angular points P, B on a straight line parallel to AC:

the two triangles PAC, BAC are equal. (Prop. 37.)

Add to each the figure $ACDEFG$;

then the figure $PCDEFG$ is equal to the figure $ABCDEFG$.

Now the sides of the figure $PCDEFG$ are fewer by one than the sides of the figure $ABCDEFG$; therefore by continued application of this process we can construct a series of figures all equal to the given figure, the sides of each figure being fewer by one than the sides of the figure last preceding.

We shall thus ultimately obtain a triangle equal to the given rectilineal figure.

It will be seen that by the method adopted in Proposition 41 A a triangle can be constructed equal to a given rectilineal figure of 4 sides by using the process once, to a figure of 5 sides by using it twice, and to a figure of n sides by using it $n-3$ times.

EXERCISES.

1. On one side of a given triangle construct an isosceles triangle equal to the given triangle.

2. On one side of a given quadrilateral construct a rectangle equal to the quadrilateral.

3. Construct a triangle equal in area to a given convex five-sided figure *ABCDE*: *AB* is to be one side of the triangle and *AE* the direction of one of the other sides.

4. Bisect a given (1) parallelogram, (2) triangle, (3) quadrilateral by a straight line drawn through a given point in one side of the figure.

5. *ABCD* is a given quadrilateral: construct a quadrilateral of equal area, having *AB* for one side, and another side on a given straight line parallel to *AB*.

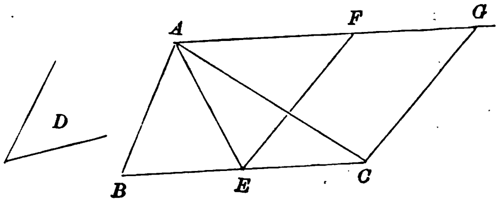
6. *ABCD* is a given quadrilateral: construct a triangle, whose base shall be in the same straight line as *AB*, its vertex at a given point *P* in *CD*, and its area equal to that of the given quadrilateral.

PROPOSITION 42.

To construct a parallelogram equal to a given triangle, and having an angle equal to a given angle.

Let ABC be the given triangle, and D the given angle: it is required to construct a parallelogram equal to ABC, and having an angle equal to D.

CONSTRUCTION. Bisect BC at E:
draw AE,
and from the point E, in the straight line EC, draw EF making the angle CEF equal to the angle D; (Prop. 23.)
through A draw AFG parallel to EC meeting EF in F,
and through C draw CG parallel to EF meeting AFG in G:
then $FECG$ is a parallelogram constructed as required.



PROOF. Because the opposite sides of the quadrilateral $FECG$ are parallel,
$FECG$ is a parallelogram.
Because the triangles ABE, AEC have the sides BE, EC equal and in a straight line, and the angular point A common,
the triangle ABE is equal to the triangle AEC;
(Prop. 38, Coroll.)
therefore the triangle ABC is double of the triangle AEC.
Because the parallelogram $FECG$ and the triangle AEC have a common side EC and the point A lies on the same straight line as the side FG, (Prop. 41.)
the parallelogram $FECG$ is double of the triangle AEC.
Therefore the parallelogram $FECG$ is equal to the triangle ABC, and it has an angle CEF equal to the given angle D.

Wherefore *a parallelogram $FECG$ has been constructed equal to the given triangle ABC, and having an angle CEF equal to the given angle D.*

EXERCISES.

1. On one side of a given triangle construct a rectangle equal to the triangle.

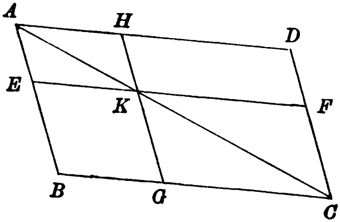
2. On one side of a given triangle construct a rhombus equal to the triangle. Is this always possible?

3. On one side of a given triangle as diagonal construct a rhombus equal to the triangle.

PROPOSITION 43.

Complements of parallelograms about a diagonal of a parallelogram, are equal.

Let $ABCD$ be a parallelogram, of which AC is a diagonal; and EH, GF are parallelograms about AC; and KB, KD the complements: (See note on page 111.)
it is required to prove that KB is equal to KD.



PROOF. Because BD is a parallelogram, and AC a diagonal,
the triangle ABC is equal to the triangle ADC. (Prop. 34.)
Now the triangle ABC is equal to the two triangles AEK, KGC and the parallelogram KB;
and the triangle ADC is equal to the two triangles AHK, KFC and the parallelogram KD.
Therefore the two triangles AEK, KGC and the parallelogram KB are together equal to the two triangles AHK, KFC and the parallelogram KD.
Again, because EH is a parallelogram and AK a diagonal,
the triangle AEK is equal to the triangle AHK; (Prop. 34.)
and because GF is a parallelogram, and KC a diagonal,
the triangle KGC is equal to the triangle KFC. (Prop. 34.)
Therefore taking away equals from equals, the remainder, the complement KB, is equal to the remainder, the complement KD.

Wherefore, *complements of parallelograms* &c.

If through a point K on a diagonal AC of a parallelogram $ABCD$, straight lines HKG, EKF be drawn parallel to the sides AB, BC respectively to meet the sides AD, BC, AB, DC in H, G, E, F respectively; then EH, GF are called **parallelograms about the diagonal** AC, and the parallelograms EG, FH are called **complements** of these parallelograms.

EXERCISES.

1. Prove that in the diagram of Proposition 43, the following are pairs of equal triangles: ABK and ADK; AEC and AHC; AKG and AKF.

2. The diagonals of parallelograms about a diagonal of a parallelogram are parallel.

3. Parallelograms about a diagonal of a square are squares.

4. If $ABCD$, $AEFG$ be two squares so placed that the angles at A coincide, then A, F, C lie on a straight line.

5. If through E a point within a parallelogram $ABCD$ straight lines be drawn parallel to AB, BC, and the parallelograms AE, EC be equal, the point E lies in the diagonal BD.

PROPOSITION 44.

To construct a parallelogram equal to a given parallelogram, having an angle equal to an angle of the given parallelogram, and having a side equal to a given straight line.

Let $ABCD$ be the given parallelogram, and EF the given straight line:

it is required to construct a parallelogram equal to $ABCD$, having an angle equal to the angle BAD, and having a side equal to EF.

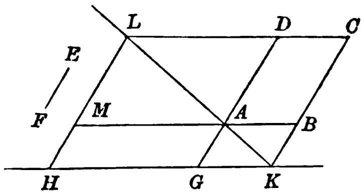
CONSTRUCTION. Produce DA to G, and make AG equal to EF. (Prop. 3.)

Through G draw HGK parallel to AB meeting CB produced in K. (Prop. 31.)

Draw KA and produce it to meet CD produced in L,

and through L draw LMH parallel to DAG to meet BA produced in M and HGK in H:

then $MAGH$ is a parallelogram constructed as required.



PROOF. Because $LCKH$ is a parallelogram, KL a diagonal, and MG, BD complements of parallelograms about the diagonal KL,

MG is equal to BD. (Prop. 43.)

Again, because the straight lines BAM, DAG intersect at A, the angle MAG is equal to the vertically opposite angle BAD. (Prop. 15.)

And AG is equal to EF. (Constr.)

Wherefore, *a parallelogram $MAGH$ has been constructed equal to the given parallelogram $ABCD$, having an angle MAG equal to the angle BAD and having a side AG equal to the given straight line EF.*

EXERCISES.

1. On a given straight line construct a rectangle equal to a given rectangle.

2. On a given straight line construct a rhombus equal to a given triangle. Is this always possible?

3. Construct a rectangle equal to the sum of two given rectangles.

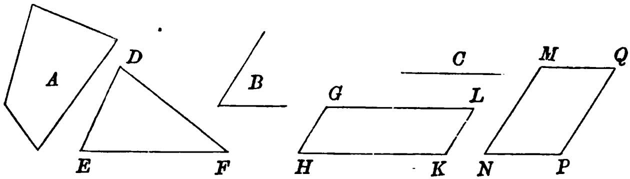
4. Construct a rectangle equal to the difference of two given rectangles.

PROPOSITION 45.

To construct a parallelogram equal to a given rectilineal figure, having a side equal to a given straight line, and having an angle equal to a given angle.

Let A be the given rectilineal figure, B the given angle, and C the given straight line:
it is required to construct a parallelogram equal to the figure A, having an angle equal to the angle B, and having a side equal to C.

CONSTRUCTION. Construct the triangle DEF equal to the figure A. (Prop. 41 A.)
Construct the parallelogram $GHKL$ equal to the triangle DEF, having the angle GHK equal to the angle B. (Prop. 42.)
Construct the parallelogram $MNPQ$ equal to the parallelogram $GHKL$, having an angle MNP equal to the angle GHK, and having the side MN equal to C. (Prop. 43.)



PROOF. Because the triangle DEF is equal to the figure A, (Constr.)
and the parallelogram GK is equal to the triangle DEF, (Constr.)
and the parallelogram MP is equal to the parallelogram GK, (Constr.)
∴ the parallelogram MP is equal to the figure A.
Because the angle GHK is equal to the angle B, (Constr.)
and the angle MNP is equal to the angle GHK, (Constr.)
∴ the angle MNP is equal to the angle B.
And MN is equal to C. (Constr.)

Wherefore *a parallelogram MNPQ has been constructed equal to the given rectilineal figure A, having the side MN equal to the given straight line C, and having the angle MNP equal to the given angle B.*

EXERCISES.

1. On a given straight line as diagonal, construct a rhombus equal to a given triangle.

2. Construct a right-angled triangle, having given the hypotenuse and the perpendicular from the right angle on it. (See Exercise 8, page 87.)

3. Construct a rectangle equal to a given rectangle, and having a diagonal equal to a given straight line.

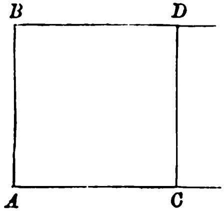
4. Construct a rectangle equal to the sum of two given triangles.

PROPOSITION 46.

On a given straight line to construct a square.

Let AB be the given straight line:
it is required to construct a square on AB.

CONSTRUCTION. From the point A draw AC at right angles to AB; (Prop. 11.)
and make AC equal to AB; (Prop. 3.)
through B draw BD parallel to AC, (Prop. 31.)
and through C draw CD parallel to AB meeting BD in D:
then $ABDC$ is a square constructed as required.



PROOF. Because CD is parallel to AB,
and BD to AC,
the figure $ABDC$ is a parallelogram. (Def. 18.)
Again the angle CAB is a right angle;
therefore the parallelogram $ABDC$ is a rectangle. (Def. 19.)
Again, the adjacent sides AC, AB are equal; (Constr.)
therefore the rectangle $ABDC$ is a square. (Def. 20.)
Wherefore, *$ABDC$ is a square constructed on the given straight line AB.*

EXERCISES.

1. If two squares be equal in area, their sides are equal.

2. The squares on two equal straight lines are equal in all respects.

3. If in the sides AB, BC, CD, DA of a square points E, F, G, H be taken so that AE, BF, CG, DH are equal: then $EFGH$ is a square.

4. If the diagonals of a quadrilateral be equal and bisect each other at right angles, the quadrilateral is a square.

5. On the sides AC, BC of a triangle ABC, squares $ACDE$, $BCFG$ are constructed: shew that the straight lines AF and BD are equal.

6. Construct a square so that one side shall be in a given straight line and two other sides shall pass through two given points.

7. Construct a square so that two opposite sides shall pass through two given points, and its diagonals intersect at a third given point.

8. Prove that the straight line, bisecting the right angle of a right-angled triangle, passes through the intersection of the diagonals of the square constructed on the outer side of the hypotenuse.

PROPOSITION 47.

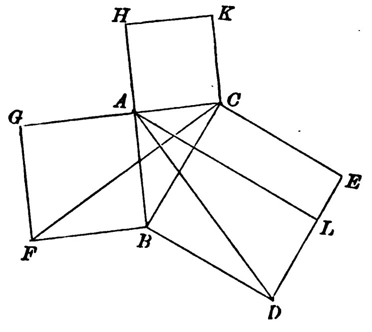
In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other sides.

Let *ABC* be a right-angled triangle, having the right angle *BAC*:
it is required to prove that the square on *BC* is equal to the sum of the squares on *BA*, *AC*.

CONSTRUCTION. On *BC* on the side away from *A* construct the square *BDEC*, (Prop. 46.)
and similarly on *BA*, *AC* construct the squares *BAGF*, *ACKH*;

through *A* draw *AL* parallel to *BD* meeting *DE* in *L*; (Prop. 31.)

and draw *AD*, *FC*.



PROOF. Because each of the angles *BAC*, *BAG* is a right angle, the two straight lines *AC*, *AG*, on opposite sides of *AB*, make with it at *A* the adjacent angles together equal to two right angles;

therefore *CA* is in the same straight line with *AG*. (Prop. 14.)

The angle *DBC* is equal to the angle *FBA*,
for each of them is a right angle. (Prop. 10 A.)

Add to each of these equals the angle ABC;
then the angle DBA is equal to the angle FBC.

And because in the triangles ABD, FBC,
AB is equal to FB and BD to BC;
and the angle ABD is equal to the angle FBC;
the triangles ABD, FBC are equal in all respects. (Prop. 4.)

Because the parallelogram BL and the triangle ABD have a common side BD and A is in the same straight line as the side of BL opposite to BD, (Prop. 41.)
the parallelogram BL is double of the triangle ABD.

And because the square GB and the triangle FBC have a common side FB, and C is in the same straight line as the side of GB opposite to FB, (Prop. 41.)
the square GB is double of the triangle FBC.

Now the doubles of equals are equal.

Therefore the parallelogram BL is equal to the square GB.

Similarly it can be proved that the parallelogram CL is equal to the square HC.

Therefore the whole square $BDEC$, which is the sum of the rectangles BL, CL, is equal to the sum of the two squares GB, HC.

And the square $BDEC$ is constructed on BC, and the squares GB, HC on BA, AC.

Therefore the square on the side BC is equal to the sum of the squares on the sides BA, AC.

Wherefore, *in a right-angled triangle* &c.

EXERCISES.

1. Construct a square equal to the difference of two given squares.

2. The diagonals of a quadrilateral intersect at right angles. Prove that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair.

3. If O be the point of intersection of the perpendiculars drawn from the angles of a triangle upon the opposite sides, the squares on OA and BC are together equal to the squares on OB and CA, and also to the squares on OC and AB.

4. Divide a given straight line into two parts so that the sum of the squares on the parts may be equal to a given square.

5. Divide a given straight line so that the difference of the squares on the parts is equal to a given square.

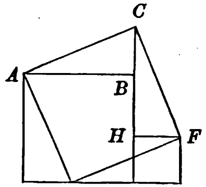
The proof of the theorem "the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other sides," which we have given in the text of the 47th proposition, is attributed to Euclid.

Tradition however says that the first person to discover a proof of the truth of the theorem was Pythagoras, a Greek philosopher who lived between 570 and 500 B.C. The theorem is in consequence often quoted as **the Theorem of Pythagoras.** What was the nature of Pythagoras' proof is not known.

The theorem is one of great importance and a large number of proofs of its truth have been discovered. It is advisable that the student should be made acquainted with some proofs besides the one given in the text.

We have made a selection of five proofs of the theorem: in each case not attempting to give the complete proof, but merely giving hints of the line of argument to be used, and leaving the student to develope it more fully.

PROOF I. Take a right-angled triangle ABC, and on the side AB away from C construct the square $ABDE$, and on the hypotenuse AC on the same side as B construct the square $ACFG$. From F draw FH perpendicular to BD, and FK perpendicular to ED produced.



It may be proved that

(1) CBD is a straight line,

(2) G lies in DE,

(3) the triangles ABC, AEG, CHF, GKF are all equal,

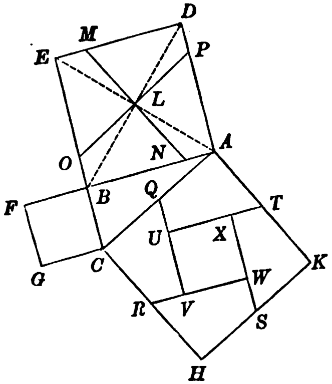
(4) HK is a square and equal to the square on BC.

PROOF II. Take a right-angled triangle *ABC* and on the sides *AB*, *BC*, *CA* away from *C*, *A*, *B* construct the squares *BADE*, *CBFG*, *ACHK*.

Through *L* the intersection of the diagonals *AE*, *BD* of the square on the larger side *AB*, draw *MLN* perpendicular to *CA* and *OLP* parallel to *CA*.

Take *Q*, *R*, *S*, *T* the middle points of the sides *AC*, *CH*, *HK*, *KA* of the square on the hypotenuse.

Through *Q*, *S* draw *QUV*, *SWX* parallel to *BC*, and through *R*, *T* draw *RVW*, *TXU* parallel to *AB*.



It may be proved that

(1) all the quadrilaterals *LMEO*, *LOBN*, *LNAP*, *LPDM*, *AQUT*, *CRVQ*, *HSWR*, *KTXS* are equal to one another,

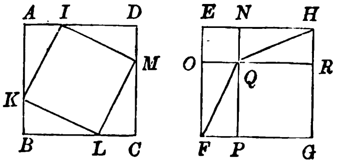
(2) the quadrilateral *UVWX* is a square,

(3) the squares *CF*, *UW* are equal.

Proof III. Take two equal squares $ABCD$, $EFGH$.

Take any point I in AD, and measure off BK, CL, DM, EN, EO each equal to AI.

Draw IK, KL, LM, MI; through N draw NQP parallel to EF, and through O draw OQR parallel to EH. Draw QF, QH.



It may be proved that

(1) the square $ABCD$ is divided into one square $IKLM$ and four equal right-angled triangles,

(2) the square $EFGH$ is divided into two squares $EOQN$, $QPGR$ and four equal triangles,

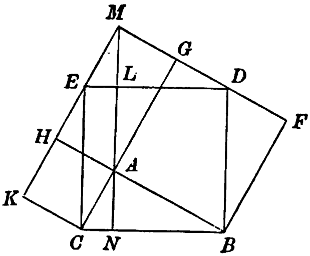
(3) all the triangles are equal to each other,

(4) the square IL is equal to the sum of the squares ON, PR,

(5) the three squares IL, ON, PR are squares on the hypotenuse and on the sides of one or other of the equal triangles.

Proof IV. Take a right-angled triangle ABC and on the hypotenuse BC on the same side as A construct the square $BCED$, and on the sides CA, AB away from B, C construct the squares $CAHK$, $ABFG$.

Through A draw $MLAN$ perpendicular to BC, and produce FG, KH to meet $MLAN$.



It may be proved that

(1) D lies in FG,

(2) E lies in KH produced,

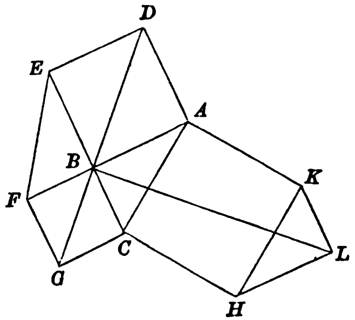
(3) the rectangle BL, and the square AF are each equal to the parallelogram AD,

(4) the rectangle CL and the square AK are each equal to the parallelogram AE.

Proof V. Take a right-angled triangle ABC: on the sides AB, BC, CA away from C, A, B construct the squares $BADE$, $CBFG$, $ACHK$.

On HK construct a triangle HLK equal in all respects to the triangle ABC having HL parallel to AB, and KL to CB.

Draw FE, GB, BD, BL.



It may be proved that

(1) GBD is a straight line.

(2) the triangles FBE, CBA are equal,

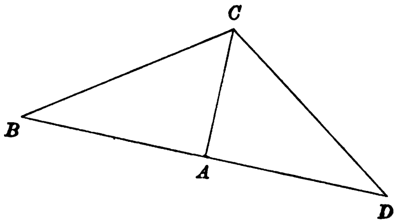
(3) all the quadrilaterals $GFED$, $GCAD$, $BCHL$, $LKAB$ are equal to one another.

PROPOSITION 48.

If the square on one side of a triangle be equal to the sum of the squares on the other sides, the angle contained by these two sides is a right angle.

Let the square on *BC*, one of the sides of the triangle *ABC*, be equal to the sum of the squares on the other sides *BA*, *AC*:
it is required to prove that the angle *BAC* is a right angle.

CONSTRUCTION. From the point *A* draw *AD* at right angles to *AC*; (Prop. 11.)
and make *AD* equal to *BA*; (Prop. 3.)
and draw *DC*.



PROOF. Because *DA* is equal to *BA*,
the square on *DA* is equal to the square on *BA*.
To each of these equals add the square on *AC*;
then the sum of the squares on *DA*, *AC* is equal to the sum of the squares on *BA*, *AC*.
Now because the angle *DAC* is a right angle,
the square on *DC* is equal to the sum of the squares on *DA*, *AC*. (Prop. 47.)
And the square on *BC* is equal to the sum of the squares on *BA*, *AC*. (Hypothesis.)
Therefore the square on *DC* is equal to the square on *BC*,
and *DC* is equal to *BC*.
And because in the triangles *DAC*, *BAC*,
DA is equal to *BA*,
AC to *AC*,
and *CD* to *CB*,
the triangles are equal in all respects; (Prop. 8.)

In Arithmetic or in Algebra, if we wish to represent a given length, we take a definite length, for instance an inch, as *a unit of length* and we express the given length by the *number* of units of length contained in it.

In the same way, if we wish to represent a given area, we take a definite area, for instance a square inch, as *a unit of area*, and we express the given area by the *number* of units of area contained in it.

It is easily seen that, if a rectangle have 2 inches in one side and 3 inches in an adjacent side, its area consists of 2×3 or 6 squares, each square having one inch as its side, and similarly that, if a rectangle have m units of length in one side and n units of length in an adjacent side, its area consists of mn squares, each square having a unit of length as its side.

Thus in Arithmetic or in Algebra the area of a rectangle is represented by the *product of the numbers*, which represent the lengths of two adjacent sides.

If the rectangle be a square, its area is represented by the *square of the number*, which represents the length of a side.

In consequence of this connection between Algebra and Geometry, there is a certain correspondence between the theorems and problems of the Second Book of Euclid, and theorems and problems in Algebra.

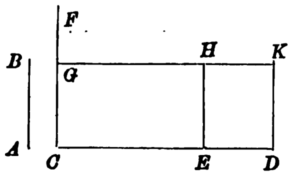
A short statement of a corresponding proposition in Algebra is given as a note to each Proposition, in which statement each straight line is represented by a corresponding letter, and each area by a corresponding product.

PROPOSITION 1.

*If there be two straight lines, one of which is divided into any two parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line, and each of the parts of the divided line**.

Let AB and CD be two straight lines; and let CD be divided into any two parts at the point E:
it is required to prove that the rectangle contained by AB, CD is equal to the sum of the rectangles contained by AB, CE, and by AB, ED.

CONSTRUCTION. From the point C draw CF at right angles to CD; (I. Prop. 11.)
and make CG equal to AB; (I. Prop. 3.)
through G draw GHK parallel to CD; (I. Prop. 31.)
and through D, E draw DK, EH parallel to CG meeting GHK in K, H.



PROOF. Each of the figures CK, CH, EK is a parallelogram, (Constr.)
and each of the figures has one angle a right angle; (I. Prop. 29, Coroll.)
therefore each of the figures is a rectangle.
Now the rectangle CK is equal to the sum of the rectangles CH, EK.

* The algebraical equivalent of this theorem is the equation
$$a(b+c)=ab+ac.$$

But CK is contained by AB, CD,
for it is contained by CG, CD, and CG is equal to AB.
And CH is contained by AB, CE,
for it is contained by CG, CE, and CG is equal to AB.
And EK is contained by AB, ED,
for it is contained by EH, ED, and EH is equal to CG, which is equal to AB.

Therefore the rectangle contained by AB, CD is equal to the sum of the rectangles contained by AB, CE, and by AB, ED.

Wherefore, *if there be two straight lines* &c.

COROLLARY 1. *If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and each of the parts of the divided line.*

If CD be divided into three parts at the points E, F:
the rectangle AB, CD is equal to the sum of the rectangles AB, CE, and AB, ED: (Prop. 1.)
and the rectangle AB, ED is equal to the sum of the rectangles AB, EF and AB, FD:
therefore the rectangle AB, CD is equal to the sum of the rectangles AB, CE; AB, EF and AB, FD.
And so on for any number of points of division.

COROLLARY 2. *If there be two straight lines, one of which is divided into any two parts, the rectangle contained by the undivided line and one of the parts of the divided line is equal to the difference of the rectangles contained by the undivided line and the whole of the divided line and by the undivided line and the remaining part of the divided line.*

EXERCISES.

1. If A, B, C, D be four points in order in a straight line, then the sum of the rectangles AB, CD, and AD, BC is equal to the rectangle AC, BD.

2. If there be two straight lines, each of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by each of the parts of the first line and each of the parts of the second line.

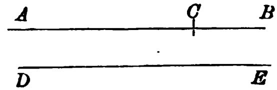
PROPOSITION 2.

If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole and each of the parts.*

Let the straight line AB be divided into any two parts at the point C:

it is required to prove that the square on AB is equal to the sum of the rectangles contained by AB, AC and by AB, CB.

CONSTRUCTION. Take DE a straight line equal to AB.



PROOF. The rectangle DE, AB is equal to the sum of the rectangles DE, AC and DE, CB. (Prop. 1.)

Because DE is equal to AB,
the rectangle DE, AB is equal to the square on AB,
the rectangle DE, AC to the rectangle AB, AC,
and the rectangle DE, CB to the rectangle AB, CB:
therefore the square on AB is equal to the sum of the rectangles AB, AC, and AB, CB.

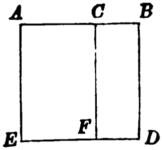
Wherefore, *if a straight line* &c.

* The algebraical equivalent of this theorem is the equation

$$(a+b)^2 = (a+b)\,a + (a+b)\,b.$$

Outline of Alternative Proof.

On AB construct the square $AEDB$,
and draw CF parallel to AE to meet ED in F.



It may be proved that
(1) AF is the rectangle AB, AC,
and (2) CD is the rectangle AB, CB,
and hence that the square on AB is equal to the sum of the rectangles AB, AC and AB, CB.

EXERCISES.

1. D is a point in the hypotenuse BC of a right-angled triangle ABC: prove that, if the rectangle BD, BC be equal to the square on AC, then the rectangle BC, DC is equal to the square on AB.

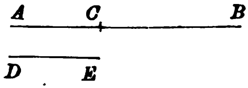
2. A point D is taken in the side BC of a triangle ABC: prove that, if the rectangles BD, BC and BC, DC be equal to the squares on AB, AC respectively, the angle BAC is a right angle.

PROPOSITION 3.

*If a straight line be divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the sum of the square on that part and the rectangle contained by the two parts**.

Let the straight line AB be divided into any two parts at the point C:
it is required to prove that the rectangle AB, AC is equal to the sum of the square on AC and the rectangle AC, CB.

CONSTRUCTION. Take DE a straight line equal to AC.



PROOF. The rectangle DE, AB is equal to the sum of the rectangles DE, AC, and DE, CB. (Prop. 1.)

Because DE is equal to AC,
the rectangle DE, AB is equal to the rectangle AC, AB,
the rectangle DE, AC to the square on AC,
and the rectangle DE, CB to the rectangle AC, CB;
therefore the rectangle AB, AC is equal to the sum of the square on AC and the rectangle AC, CB.

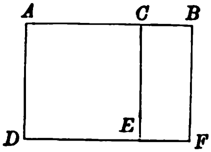
Wherefore, *if a straight line* &c.

* The algebraical equivalent of this theorem is the equation

$$(a+b)\,a=a^2+ab.$$

Outline of Alternative Proof.

On AC construct the square $ADEC$, and draw BF parallel to AD to meet DE produced in F.



It may be proved that

(1) AF is the rectangle AB, AC,

and (2) CF is the rectangle AC, CB,

and hence that the rectangle AB, AC is equal to the sum of the square on AC and the rectangle AC, CB.

EXERCISES.

1. AD is drawn perpendicular to the hypotenuse BC of a right-angled triangle ABC: prove that the rectangle BD, DC is equal to the square on AD.

2. In the triangle ABC, AD is the perpendicular from A on BC: prove that, if the rectangle BD, DC be equal to the square on AD, the angle A is a right angle.

3. In the triangle ABC, AD is the perpendicular from A on BC: prove that, if the rectangle BD, BC be equal to the square on AB, the angle A is a right angle.

PROPOSITION 4.

If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the parts and twice the rectangle contained by the parts.*

Let the straight line AB be divided into any two parts at the point C:

it is required to prove that the square on AB is equal to the sum of the squares on AC and CB, and twice the rectangle contained by AC, CB.

PROOF. Because AB is divided into two parts at C, the square on AB is equal to the sum of the rectangles AB, AC and AB, CB, (Prop. 2.)

and the rectangle AB, AC is equal to the sum of the square on AC and the rectangle AC, CB, (Prop. 3.)

and the rectangle AB, CB is equal to the sum of the square on CB and the rectangle AC, CB. (Prop. 3.)

Therefore the square on AB is equal to the sum of the squares on AC and CB, and twice the rectangle AC, CB.

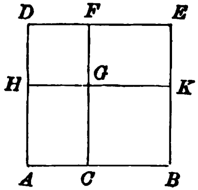
Wherefore, *if a straight line* &c.

* The algebraical equivalent of this theorem is the equation

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

Outline of Alternative Proof.

On AB construct the square $ADEB$:
draw CGF parallel to AD: make DH equal to AC,
and draw HGK parallel to AB.



It may be proved that

(1) DG is equal to the square on AC,

(2) GB is the square on CB,

and (3) HC, FK are each equal to the rectangle AC, CB,

and hence that the square on AB is equal to the sum of the squares on AC, CB and twice the rectangle AC, CB.

EXERCISES.

1. The square on a straight line is four times the square on half of the line.

2. If the sides BC, CA, AB of a right-angled triangle ABC be bisected in the points D, E, F respectively, twice the squares on AD, BE and CF are together equal to three times the square on the hypotenuse.

3. If, in an acute-angled triangle ABC, AD be drawn perpendicular to BC, then the sum of the squares on AB, AC and twice the rectangle BD, DC is equal to the sum of the square on BC and twice the square on AD.

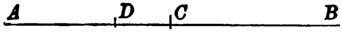
4. If BAC be an obtuse angle and BD, CE be drawn at right angles to CA, BA respectively, then the rectangle BA, AE is equal to the rectangle CA, AD.

5. $ABCD$ is a square and E, F, G, H are points on the sides AB, BC, CD, DA respectively: prove that, if $EFGH$ be a rectangle, it is either double of the rectangle AE, EB, or equal to the sum of the squares on AE, EB.

PROPOSITION 5.

If a straight line be divided into two equal parts and also into two unequal parts, the sum of the rectangle contained by the unequal parts and the square on the line between the points of section is equal to the square on half the line.*

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D: it is required to prove that the sum of the rectangle AD, DB and the square on DC, is equal to the square on AC.



PROOF. Because DB is divided into two parts at C,
the rectangle AD, DB is equal to the sum of the
rectangles AD, DC and AD, CB, (Prop. 1.)
that is to the sum of the rectangles AD, DC, and AD, AC,
since AC is equal to CB. (Hypothesis.)

And because AC is divided into two parts at D,
the rectangle AD, AC is equal to the sum of the square on AD and the rectangle AD, DC. (Prop. 3.)
Therefore the rectangle AD, DB is equal to the sum of the square on AD and twice the rectangle AD, DC.

Add to each of these equals the square on DC;
then the sum of the square on DC and the rectangle AD, DB is equal to the sum of the squares on AD, DC and twice the rectangle AD, DC, which sum is equal to the square on AC. (Prop. 4.)
Therefore the rectangle AD, DB, together with the square on DC, is equal to the square on AC.

Wherefore, *if a straight line* &c.

* The algebraical equivalent of this theorem is the equation

$$(a-b)(a+b)+b^2=a^2.$$

The theorems of Propositions 5 and 6 may both be included in one enunciation, thus, *The difference of the squares on two given straight lines is equal to the rectangle contained by the sum and the difference of the lines.*

The straight lines in both propositions are AD, BD: the only difference being that in Prop. 5. BD is the greater and in Prop. 6. AD is the greater.

For an outline of an alternative proof of Propositions 5 and 6, see page 147.

EXERCISES.

1. A straight line is divided into two parts; shew that, if twice the rectangle of the parts be equal to the sum of the squares on the parts, the straight line is bisected.

2. Divide a given straight line into two parts such that the rectangle contained by them shall be the greatest possible.

3. Divide a given straight line into two parts such that the sum of the squares on the two parts may be the least possible.

4. Divide a given straight line into three parts so that the sum of the squares on them may be the least possible.

5. ABC is an equilateral triangle and D is any point in the side BC. Prove that the square on BC is equal to the rectangle contained by BD, DC, together with the square on AD.

6. A point D is taken on the hypotenuse BC of a right-angled triangle ABC; prove that, if the rectangle BD, DC be equal to the square on AD, D is either the middle point of BC or the foot of the perpendicular from A on BC.

PROPOSITION 6.

If a straight line be bisected, and produced to any point, the sum of the rectangle contained by the whole line thus produced and the part of it produced, and the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.*

Let the straight line AB be bisected at the point C, and produced to the point D:
it is required to prove that the sum of the rectangle AD, BD, and the square on CB is equal to the square on CD.

Proof. Because AD is divided into two parts at B, the rectangle AD, BD is equal to the sum of the rectangle AB, BD and the square on BD. (Prop. 3.)

Because AB is bisected in C,
the rectangle AB, BD is double of the rectangle CB, BD; (Prop. 1.)
therefore the rectangle AD, BD is equal to the sum of the square on BD and twice the rectangle CB, BD.

Add to each of these equals the square on CB;
then the sum of the square on CB and the rectangle AD, BD is equal to the sum of the squares on CB, BD and twice the rectangle CB, BD.
And the sum of the squares on CB, BD and twice the rectangle CB, BD is equal to the square on CD; (Prop. 4.)
therefore the sum of the rectangle AD, BD and the square on CB is equal to the square on CD.

Wherefore, *if a straight line* &c.

* The algebraical equivalent of this theorem is the equation

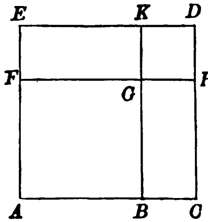
$$(a+b)(b-a)+a^2=b^2,$$

or

$$(2a+b)\,b+a^2=(a+b)^2.$$

Outline of Alternative Proof of Propositions 5 and 6.

Let *A*, *B*, *C* be any three points in a straight line. Draw *AFE*, *BGK*, *CHD* at right angles to *ABC*.



Take *AF* equal to *AB* and *FE* to *BC*, and draw *EKD*, *FGH* parallel to *ABC*.

It may be proved that

(1) *FB* is the square on *AB*,

(2) *KH* is equal to the square on *BC*,

(3) *EB* is the rectangle *AB*, *AC*,

and (4) *EH* is the rectangle *AC*, *BC*,

and hence that the difference of the squares on *AB*, *BC*, which is equal to the difference of the rectangles *EB*, *EH*, is equal to the rectangle contained by the sum and the difference of *AB* and *BC*.

EXERCISES.

1. A straight line is divided into two equal and also into two unequal parts; prove that the difference of the squares on the two unequal parts is equal to twice the rectangle contained by the whole line and the part between the points of section.

2. The straight line *AB* is bisected in *C* and produced to *D*, *CE* is drawn perpendicular to *AB* and equal to *BD*, and a point *F* is taken in *BD* so that *EF* is equal to *CD*; prove that the rectangle *DF*, *DA* together with the rectangle *DF*, *FB* is equal to the square on *BD*.

PROPOSITION 7.

*If a straight line be divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to the sum of twice the rectangle contained by the whole line and that part and the square on the other part**.

Let the straight line AB be divided into any two parts at the point C:
it is required to prove that the sum of the squares on AB, CB is equal to the sum of twice the rectangle AB, CB, and the square on AC.

PROOF. Because AB is divided into two points at C, the square on AB is equal to the sum of the squares on AC, CB, and twice the rectangle AC, CB. (Prop. 4.)

Add to each of these equals the square on CB;
then the sum of the squares on AB, CB is equal to the sum of the square on AC, twice the square on CB, and twice the rectangle AC, CB.

But the sum of the square on CB and the rectangle AC, CB is equal to the rectangle AB, CB. (Prop. 3.)

Therefore the sum of the squares on AB, CB, is equal to the sum of twice the rectangle AB, CB, and the square on AC.

Wherefore, *if a straight line* &c.

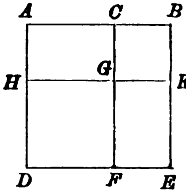
COROLLARY. *If a straight line be divided into any two parts, the square on one of the parts is less than the sum of the squares on the whole line and the other part by twice the rectangle contained by the whole line and the second part.*

* The algebraical equivalent of this theorem is the equation

$$(a+b)^2+b^2=2(a+b)\,b+a^2.$$

Outline of Alternative Proof.

On AB construct the square $ADEB$
and draw CGF parallel to AD.
Take BK equal to BC,
and draw KGH parallel to BCA.



It may be proved that

(1) AK, CE are each equal to the rectangle AB, CB,

(2) HF is equal to the square on AC,

and (3) CK is the square on CB,

and hence that the sum of the squares on AB, CB is equal to the sum of twice the rectangle AB, CB and the square on AC.

EXERCISES.

1. $ACDB$ is a straight line, and D bisects CB: prove that the square on AC is less than the sum of the squares on AD, DB by twice the rectangle AD, DB.

2. If BAC be an acute angle and BD, CE be drawn perpendicular to CA, AB respectively, then the rectangle BA, AE is equal to the rectangle CA, AD.

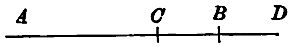
3. Shew how to divide a given straight line into two parts such that the difference of the squares described on them may be equal to a given rectangle. Is a solution always possible?

PROPOSITION 8.

*If a straight line be divided into any two parts, the sum of the square on one part and four times the rectangle contained by the whole line and the other part, is equal to the square on the straight line which is made up of the whole and the second part**.

Let the straight line AB be divided into any two parts at the point C:
it is required to prove that the sum of four times the rectangle AB, CB, and the square on AC is equal to the square on the straight line made up of AB and CB together.

CONSTRUCTION. Produce AB to D,
and make BD equal to CB. (I. Prop. 3.)



PROOF. Because AD is divided into two parts at B,
the sum of the squares on AB, BD and twice the rectangle AB, BD, is equal to the square on AD: (Prop. 4.)
and because AB is divided into two parts at C,
the sum of the square on AC and twice the rectangle AB, CB,
is equal to the sum of the squares on AB, CB. (Prop. 7.)
Add these equals together;
then the sum of the squares on AB, BD, AC and four times the rectangle AB, CB,
is equal to the sum of the squares on AB, CB, AD.
Take away from these equals, the equals the sum of the squares on AB, BD and the sum of the squares on AB, CB;
then the sum of the square on AC and four times the rectangle AB, CB is equal to the square on AD.

Wherefore, *if a straight line* &c.

* The algebraical equivalent of this theorem is the equation

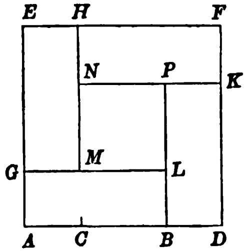
$$(a-b)^2+4ab=(a+b)^2,$$

or

$$a^2+4(a+b)b=(a+2b)^2.$$

Outline of Alternative Proof.

Produce AB to D and make BD equal to AC.
On AD construct the square $AEFD$.



Take AG, EH, FK each equal to AC, and draw BLP, MNH parallel to AE and GML, NPK parallel to AD.

It may be proved that

(1) AL, BK, FN, EM are each equal to the rectangle AB, AC,

and (2) MP is equal to the square on CB,

and hence that the sum of the square on AC and four times the rectangle AB, CB is equal to the square on AD.

EXERCISES.

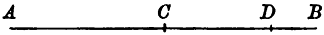
1. Prove that the square on a straight line is nine times the square on one third of the line.

2. If a straight line be bisected and produced to any point, the square on the whole line thus produced is equal to the square on the part produced together with twice the rectangle contained by the line and the line made up of the half and the part produced.

PROPOSITION 9.

If a straight line be divided into two equal, and also into two unequal parts, the sum of the squares on the two unequal parts is double of the sum of the squares on half the line and on the line between the points of section.*

Let the straight line AB be divided into two equal parts at the point C, and into two unequal parts at the point D:
it is required to prove that the sum of the squares on AD, DB is double of the sum of the squares on AC, CD.



PROOF. Because AD is divided at C,
the square on AD is equal to the sum of the squares on AC, CD and twice the rectangle AC, CD. (Prop. 4.)

And because CB is divided at D,
the sum of the square on DB and twice the rectangle CB, CD is equal to the sum of the squares on CB, CD. (Prop. 7.)

Add these pairs of equals;
then the sum of the squares on AD, DB and twice the rectangle CB, CD is equal to the sum of the squares on AC, CD, CB, CD and twice the rectangle AC, CD.
Take away from these equals twice the rectangle CB, CD, and twice the rectangle AC, CD, which are equal;
then the sum of the squares on AD, DB is equal to the sum of the squares on AC, CD, CB, CD,
that is, is equal to twice the sum of the squares on AC, CD.

Wherefore, *if a straight line* &c.

* The algebraical equivalent of this theorem is the equation
$$(a+b)^2+(a-b)^2=2(a^2+b^2).$$

The theorems of Propositions 9 and 10 may both be included in one enunciation: thus, *The sum of the squares on the sum and on the difference of two given straight lines is equal to twice the sum of the squares on the lines.*

The straight lines in both propositions are AC, CD: the only difference being that in Prop. 9. AC is the greater, and in Prop. 10. CD is the greater.

For an outline of an alternative proof of Propositions 9 and 10, see page 155.

EXERCISES.

1. A straight line is divided into two parts, such that the diagonal of the square on one of these parts is equal to the whole line. If a square be constructed whose side is the difference between the aforesaid part and half the given line, its diagonal is equal to the other of the two parts into which the line is divided.

2. Deduce a proof of II. 9 from the result of II. 5.

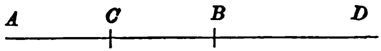
3. If a straight line be divided into two equal and also into two unequal parts, the squares on the two unequal parts are equal to twice the rectangle contained by the two unequal parts together with four times the square on the line between the points of section.

PROPOSITION 10.

If a straight line be bisected and produced to any point, the sum of the squares on the whole line thus produced and on the part produced is double of the sum of the squares on half the line and on the line made up of the half and the part produced.*

Let the straight line AB be bisected at C, and produced to D:

it is required to prove that the sum of the squares on AD, BD is double of the sum of the squares on AC, CD.



PROOF. Because AD is divided at C,

the square on AD is equal to the sum of the squares on AC, CD and twice the rectangle AC, CD; (Prop. 4.)

and because CD is divided at B,

the sum of the square on BD and twice the rectangle CB, CD is equal to the sum of the squares on CB, CD. (Prop. 7).

Add these equals;

then the sum of the squares on AD, BD and twice the rectangle CB, CD is equal to the sum of the squares on AC, CD, CB, CD and twice the rectangle AC, CD.

Take away from these equals twice the rectangle CB, CD, and twice the rectangle AC, CD, which are equal;

then the sum of the squares on AD, BD is equal to the sum of the squares on AC, CD, CB, CD,

that is, is equal to twice the sum of the squares on AC, CD.

Wherefore, *if a straight line* &c.

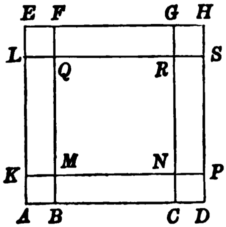
* The algebraical equivalent of this theorem is the equation

$$(a+b)^2+(b-a)^2=2(a^2+b^2).$$

Outline of Alternative Proof of Propositions 9 *and* 10.

In a straight line *ABCD*, take *AB* equal to *CD*. Through *A*, *B*, *C*, *D* draw *AE*, *BF*, *CG*, *DH* at right angles to *ABCD*.

Take *AK* equal to *AB*, *KL* to *BC* and *LE* to *AB*. Draw *EFGH*, *LQRS*, *KMNP* parallel to *ABCD*.



It may be proved that

(1) *EQ*, *ND* are each equal to the square on *AB*,

(2) *LC*, *FP* are each equal to the square on *AC*,

(3) *QN* is equal to the square on *BC*,

(4) *ED* is the square on *AD*,

and (5) the sum of *ED* and *QN* is equal to the sum of *EQ*, *ND*, *LC*, and *FP*,

and hence that the sum of the squares on *AD*, *BC* (which are the sum and the difference of *AC* and *AB*) is equal to twice the sum of the squares on *AC*, *AB*.

EXERCISES.

1. In *AB* the diameter of a circle take two points *C* and *D* equally distant from the centre, and from any point *E* in the circumference draw *EC*, *ED*: shew that the squares on *EC* and *ED* are together equal to the squares on *AC* and *AD*.

2. If in *BC* the base of a triangle a point *D* be taken such that the squares on *AB* and *BD* are together equal to the squares on *AC* and *CD*, then the middle point of *AD* will be equally distant from *B* and *C*.

3. A square *BDEC* is described on the hypotenuse *BC* of a right-angled triangle *ABC*: shew that the squares on *DA* and *AC* are together equal to the squares on *EA* and *AB*.

4. *AB* is divided into any two parts in *C*, and *AC*, *BC* are bisected in *D*, *E*: prove that the square on *AE* and three times the square on *BE* are equal to the square on *BD* and three times the square on *AD*.

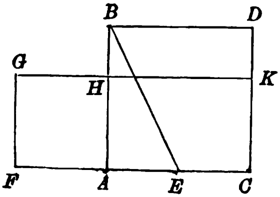
PROPOSITION 11.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one part may be equal to the square on the other part.*

Let AB be the given straight line:
it is required to divide it into two parts in a point H, so that the rectangle contained by the whole line AB and a part HB may be equal to the square on the other part AH.

CONSTRUCTION. On AB construct the square $ABDC$; (I. Prop. 46.)
bisect AC at E; (I. Prop. 10.)
draw BE; produce CA to F,
and make EF equal to EB; (I. Prop. 3.)
and on AF construct the square $AFGH$: (I. Prop. 46.)
then AB is divided at H so that the rectangle AB, HB is equal to the square on AH.

Produce GH to meet CD at K.



PROOF. Because AC is bisected at E, and produced to F, the sum of the rectangle FC, FA, and the square on AE is equal to the square on FE. (Prop. 6.)

But FE is equal to EB. (Constr.)

Therefore the sum of the rectangle FC, FA, and the square on AE is equal to the square on EB.

But, because the angle EAB is a right angle,
the square on EB is equal to the sum of the squares on AE, AB. (I. Prop. 47.)

* The algebraical equivalent of this problem is to find the smaller root of the quadratic equation $ax=(a-x)^2$, or the positive root of the quadratic equation $a(a-x)=x^2$.

Therefore the sum of the rectangle *FC*, *FA*, and the square on *AE*, is equal to the sum of the squares on *AE*, *AB*.

Take away from each of these equals the square on *AE*; then the rectangle *FC*, *FA* is equal to the square on *AB*.

But the figure *FK* is the rectangle contained by *FC*, *FA*,
for *FG* is equal to *FA*; (Constr.)
and *AD* is the square on *AB*;
therefore *FK* is equal to *AD*.

Take away from these equals the common part *AK*;
then *FH* is equal to *HD*.

But *HD* is the rectangle contained by *AB*, *HB*,
for *AB* is equal to *BD*; (Constr.)
and *FH* is the square on *AH*;
therefore the rectangle *AB*, *HB* is equal to the square on *AH*.

Wherefore *the straight line AB is divided at H, so that the rectangle AB, HB is equal to the square on AH.*

EXERCISES.

1. Shew that in a straight line divided as in II. 11 the rectangle contained by the sum and the difference of the parts is equal to the rectangle contained by the parts.

2. If the greater segment of the line divided in this proposition be divided in the same manner, the greater segment of the greater segment is equal to the smaller segment of the original line.

3. Prove that when a straight line is divided as in this proposition the square on the line made up of the given line and the smaller part is equal to five times the square on the larger part.

4. Prove that in the figure of this proposition the squares on *AB*, *HB* are together equal to three times the square on *AH*, and that the difference of the squares on *AB*, *AH* is equal to the rectangle *AH*, *AB*.

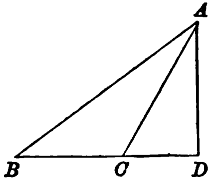
5. Produce a given straight line, so that the rectangle contained by the whole line thus produced and the given line may be equal to the square on the part produced.

PROPOSITION 12.

In an obtuse-angled triangle, if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side opposite the obtuse angle is greater than the sum of the squares on the other sides, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the part of the produced side intercepted between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, and let the angle ACB be the obtuse angle; from the point A let AD be drawn perpendicular to BC produced:

it is required to prove that the square on AB is greater than the sum of the squares on AC, CB, by twice the rectangle BC, CD.



PROOF. Because BD is divided into two parts at C, the square on BD is equal to the sum of the squares on BC, CD, and twice the rectangle BC, CD. (Prop. 4.)

To each of these equals add the square on DA;

then the sum of the squares on BD, DA is equal to the sum of the squares on BC, CD, DA, and twice the rectangle BC, CD.

But, because the angle at D is a right angle,

the square on BA is equal to the sum of the squares on BD, DA,

and the square on CA is equal to the sum of the squares on CD, DA. (I. Prop. 47.)

Therefore the square on BA is equal to the sum of the squares on BC, CA, and twice the rectangle BC, CD;

that is, the square on BA is greater than the sum of the squares on BC, CA by twice the rectangle BC, CD.

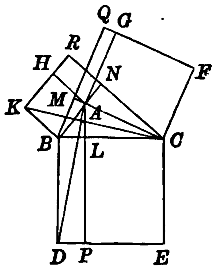
Wherefore *in an obtuse-angled triangle* &c.

Outline of Alternative Proof.

On the sides BC, CA, AB of an obtuse-angled triangle ABC, in which the angle BAC is obtuse, construct the squares $BDEC$, $CFGA$, $AHKB$.

Draw AL, BM, CN perpendicular to BC, CA, AB and produce them to meet the opposite sides (produced if necessary) of the squares in P, Q, R.

Draw AD, CK.



It may be proved that

(1) the triangle ABD is equal to the triangle KBC in all respects,

and (2) the rectangle BP is equal to the rectangle BR, and similarly that CQ is equal to CP, and AR to AQ,

and hence that the square on BC is greater than the sum of the squares on CA, AB by twice the rectangle AR or AQ.

EXERCISES.

1. The sides of a triangle are 10, 12, 15 inches: prove that it is acute-angled.

2. On the side BC of any triangle ABC, and on the side of BC remote from A, a square $BDEC$ is constructed. Prove that the difference of the squares on AB and AC is equal to the difference of the squares on AD and AE.

3. C is the obtuse angle of a triangle ABC, and D, E the feet of the perpendiculars drawn from A, B respectively to the opposite sides produced: prove that the square on AB is equal to the sum of the rectangles contained by BC, BD and AC, AE.

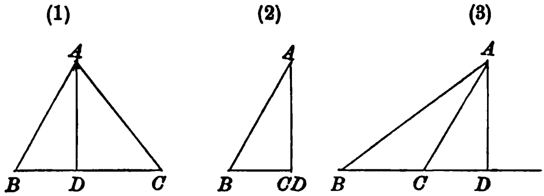
4. ABC is a triangle having the sides AB and AC equal; AB is produced beyond the base to D so that BD is equal to AB; shew that the square on CD is equal to the square on AB, together with twice the square on BC.

PROPOSITION 13.

In any triangle, the square on a side subtending an acute angle, is less than the sum of the squares on the other sides, by twice the rectangle contained by either of these sides, and the part of the side intercepted between the perpendicular let fall on it from the opposite angle, and the acute angle.

Let ABC be a triangle, and let the angle ABC be an acute angle; let AD be drawn perpendicular to BC and meet it (produced if necessary) in D:
it is required to prove that the square on AC is less than the sum of the squares on AB, BC by twice the rectangle BC, BD.

Either (1) D lies in BC, or (2) D coincides with C, or (3) D lies in BC produced.



Proof. Because fig. (1) BC is divided in D, fig. (2) D is the same point as C, or fig. (3) BD divided in C,
the sum of the squares on BC, BD is equal to the sum of the square on CD, and twice the rectangle BC, BD.
(I. Prop. 47 and II. Prop. 7.)

To each of these equals add the square on DA;
then the sum of the squares on BC, BD, DA is equal to the sum of the squares on CD, DA and twice the rectangle BC, BD.

But because the angle BDA is a right angle,
the square on AB is equal to the sum of the squares on BD, DA,
and the square on AC is equal to the sum of the squares on CD, DA. (I. Prop. 47.)
Therefore the sum of the squares on BC, AB is equal to the sum of the square on AC and twice the rectangle BC, BD:
that is, the square on AC is less than the sum of the squares on AB, BC by twice the rectangle BC, BD.

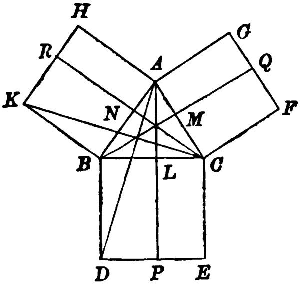
Wherefore, *in any triangle* &c.

Outline of Alternative Proof.

On the sides BC, CA, AB of an acute-angled triangle ABC construct the squares $BDEC$, $CFGA$, $AHKB$.

Draw AL, BM, CN perpendicular to BC, CA, AB, and produce them to meet the opposite sides of the squares in P, Q, R.

Draw AD, CK.



It may be proved that

(1) the triangle ABD is equal to the triangle KBC in all respects,

and (2) the rectangle BP is equal to the rectangle BR, and similarly that CQ is equal to CP, and AR to AQ,

and hence that the square on BC is less than the sum of the squares on CA, AB by twice the rectangle AQ or AR.

EXERCISES.

1. In any triangle the sum of the squares on the sides is equal to twice the square on half the base together with twice the square on the straight line drawn from the vertex to the middle point of the base.

2. The base of a triangle is given: find the locus of the vertex, when the sum of the squares on the two sides is given.

3. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

4. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

5. The squares on the sides of a quadrilateral are together greater than the squares on its diagonals by four times the square on the straight line joining the middle points of its diagonals.

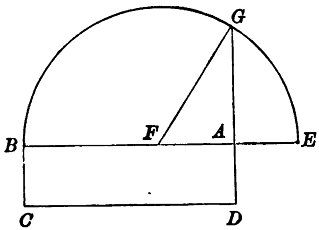
PROPOSITION 14.

*To find the side of a square equal to a given rectangle.**

Let $ABCD$ be the given rectangle :
it is required to find the side of a square equal to $ABCD$.

CONSTRUCTION. If two adjacent sides BA, AD be equal, the rectangle is a square, and BA or AD is the line required.

But if they be not equal,
produce one of them BA to E,
and make AE equal to AD; (I. Prop. 3.)
bisect BE at F; (I. Prop. 10.)
and with F as centre and FB as radius,
describe the circle BGE,
and produce DA to meet the circle in G;
then AG is the line required.
Draw FG.



PROOF. Because BE is divided into two equal parts at F, and into two unequal parts at A,
the sum of the rectangle BA, AE and the square on FA is equal to the square on FE. (Prop. 5.)

But FE is equal to FG.

Therefore the sum of the rectangle BA, AE and the square on FA is equal to the square on FG.

* The algebraical equivalent of this problem is to find the positive root of the quadratic equation $x^2 = ab$.

BOOK III.

DEFINITIONS.

Definition 1. *Any part of a circle is called* **an arc.**

The line which has been defined (I. Def. 22) as **a circle** is often spoken of as **the circumference of** the circle.

The reason of this is that a circle is defined in many books as the part of the plane contained by the line, which is then called the circumference.

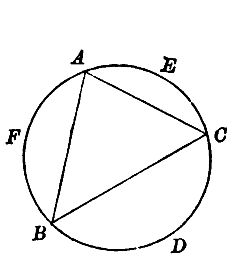
Half of a circle is called **a semicircle.**

It will be proved hereafter that a diameter bisects a circle, i.e. divides it into two equal arcs. (See page 175.)

Definition 2. *A straight line joining two points on a circle is called* **a chord of the circle.**

The straight line joining the extremities of an arc is called **the chord of the arc.**

The figure formed of an arc and the chord of the arc is called **a segment of the circle.**



In the diagram the straight lines *AB*, *BC*, *CA* are **chords** of the circle *ABC*; *AFB*, *BDC*, *CEA* are **arcs.**

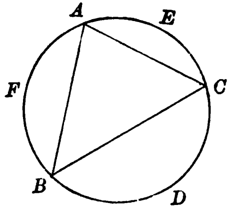
The straight line *AB* is the chord of the arc *AFB*, and it is also the chord of the arc *ACB*.

The figure formed of the arc *BFEC* and the chord *BC* is called the segment *BFEC* or *BFAC*, or more often *BFC* or *BEC* or *BAC*; and the figure formed of the arc *BDC* and the chord *BC* is called the segment *BDC*.

DEFINITION 3. *The angle contained by two chords joining a point in an arc of a circle to the extremities of the arc is called* **an angle in the arc,** *and the arc is said* **to contain the angle.**

An angle in an arc is often spoken of as **an angle in the segment** formed by the arc and the chord of the arc, and the segment is said to **contain** the angle.

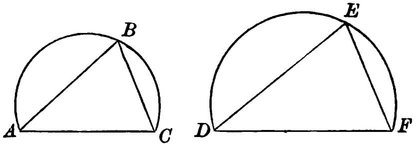
The angle *BAC* is said to be an angle in the arc (or in the segment) *BFC* and the angle *ACB* an angle in the arc (or in the segment) *ADB*; and the arc (or the segment) *BFC* is said to **contain** the angle *BAC*, and the arc (or the segment) *ADB* to contain the angle *ACB*.



An angle in an arc is said **to stand on** *the arc which forms the remainder of the circle.*

The angle *BAC* is said to stand on the arc *BDC*, and the angle *ABC* on the arc *AEC*.

DEFINITION 4. *Arcs, which contain equal angles, are said to be* **similar**; *and likewise segments, which contain equal angles, are said to be* **similar.**



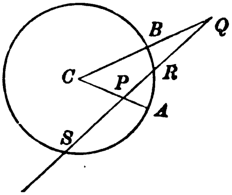
In the diagram the arcs *ABC*, *DEF* are said to be similar, when the contained angles *ABC*, *DEF* are equal; and also the segments *ABC*, *DEF* are said to be similar when the contained angles *ABC*, *DEF* are equal.

DEFINITION 5. *A point, whose distance from the centre of a circle is less than the radius of the circle, is said to be* **within the circle**; *and a point, whose distance from the centre of a circle is greater than the radius of the circle, is said to be* **without the circle.**

In the diagram the point P is within the circle, its distance CP from the centre C being less than the radius CA, and the point Q is without the circle, its distance CQ from the centre C being greater than the radius CB.

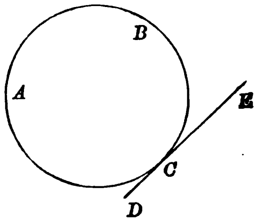
It is clear that any line drawn from a point P within a circle to a point Q without the circle must intersect the circle once at least (I. Postulate 7, page 14).

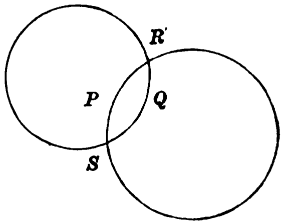
In the diagram the straight line PQ meets the circle in the points R and S, and the straight line and the circle intersect at each of those points.



DEFINITION 6. *A straight line and a circle, which pass through a point but do not intersect there, are said* **to touch one another** *at the point. The straight line is called* **a tangent to the circle.**

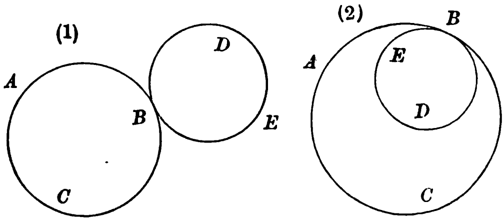
In the diagram the circle ABC and the straight line DCE pass through the point C, but do not intersect there: they touch at the point C, and DE is a tangent to the circle at the point C.





In the diagram the circles *PRS*, *QRS* meet at the points *R* and *S*, and the circles intersect at each of those points: for instance, points on the circle *PRS* near *R* on one side of *R* lie *within* the circle *QRS* and on the other side of *R* *without* it.

DEFINITION 7. *Two circles, which pass through a point but do not intersect there, are said* **to touch one another** *at the point.*



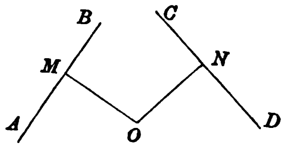
In each of the figures in the diagram the circles *ABC*, *BDE* pass through the point *B*, but do not intersect there: all points on the circle *ABC* near *B* lie without the circle *BDE*: and all points on the circle *BDE* near *B* in figure (1) lie without the circle *ABC*, and in figure (2) lie within the circle *ABC*.

(See remarks on the contact of circles on pages 199 and 201.)

DEFINITION 8. *Circles which have the same point for a centre are said to be* **concentric.**

DEFINITION 9. *The perpendicular drawn to a straight line from a point is called* **the distance of the straight line from the point.**

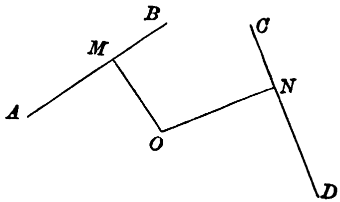
If the distances of two straight lines from a point be equal, the straight lines are said to be **equidistant from the point.**



In the diagram, if the straight lines *OM*, *ON* be drawn from the point *O* perpendicular to the two straight lines *AB*, *CD*, then *OM*, *ON* are called the distances of the two straight lines *AB*, *CD* from the point *O*.

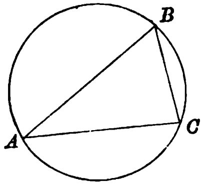
If *OM*, *ON* be equal, the straight lines *AB*, *CD* are said to be equidistant from the point *O*.

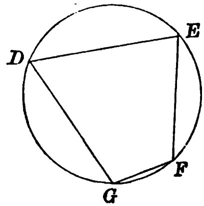
If the distances of two straight lines from a point be unequal, the line the distance of which is the longer is said to be **farther from the point,** *and the line the distance of which is the shorter is said to be* **nearer to the point.**



In the diagram, if the straight lines *OM*, *ON* be drawn from the point *O* perpendicular to the two straight lines *AB*, *CD*, and if *OM* be less than *ON*, then *CD* is said to be farther from the point *O* than *AB*, and *AB* is said to be nearer to the point *O* than *CD*.

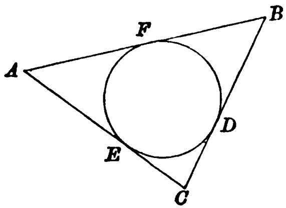
DEFINITION 10. *If all the angular points of a rectilineal figure lie on a circle, the figure is said to be* **inscribed in the circle,** *and the circle is said to be* **described about** *the figure.*

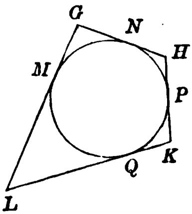




In the diagram, if the angular points of the triangle *ABC* lie on the circle *ABC*, the triangle *ABC* is said to be inscribed in the circle *ABC*, and the circle *ABC* is said to be described about the triangle *ABC*. Similarly, if the angular points *D*, *E*, *F*, *G* of the quadrilateral *DEFG* lie on the circle *DEFG*, the quadrilateral *DEFG* is said to be inscribed in the circle *DEFG*, and the circle *DEFG* is said to be described about the quadrilateral *DEFG*.

If all the sides of a rectilineal figure touch a circle, the figure is said to be **described about the circle,** *and the circle is said to be* **inscribed in** *the figure.*





In the diagram, if the sides *BC*, *CA*, *AB* of the triangle *ABC* touch the circle *DEF* at the points *D*, *E*, *F* respectively, the triangle *ABC* is

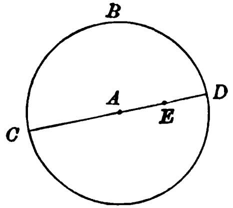
said to be described about the circle *DEF*, and the circle *DEF* is said to be inscribed in the triangle *ABC*. Similarly, if the sides *LG*, *GH*, *HK*, *KL* of the quadrilateral *GHKL* touch the circle *MNPQ* at the points *M*, *N*, *P*, *Q* respectively, the quadrilateral *GHKL* is said to be described about the circle *MNPQ*, and the circle *MNPQ* is said to be inscribed in the quadrilateral *GHKL*.

PROPOSITION 1.

A circle cannot have more than one centre.

Let A be a centre of the given circle BCD:
it is required to prove that no other point can be a centre of the circle BCD.

CONSTRUCTION. Take any point E within the circle and draw AE,
and produce AE both ways to meet the circle, beyond A in C, and beyond E in D.



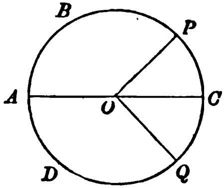
PROOF. Because A is a centre of the circle,
AC is equal to AD. (I. Def. 22.)
Now EC is greater than AC, which is only a part of it,
and ED, which is only a part of AD, is less than AD;
therefore EC is greater than ED.
But a centre of a circle is a point from which all straight lines drawn to the circle are equal; (I. Def. 22.)
therefore E cannot be a centre of the circle.
Similarly it can be proved that no point other than A can be a centre.

Wherefore, *a circle cannot have* &c.

The definition of a circle implies that the figure has a centre (I. Def. 22): it is here proved that it cannot have more than one centre: we shall therefore for the future speak of **the centre** of a circle.

PROPOSITION 1 A.

A diameter bisects a circle.



Let $ABCD$ be a circle, O the centre and AOC a diameter:
it is required to prove that the arcs ABC, ADC are equal.

CONSTRUCTION. Draw any two radii OP, OQ making equal angles POC, QOC with OC. (I. Prop. 23.)

PROOF. It is possible to shift the figure $AOCQD$ by turning it over so that AOC is not shifted,
and so that the arcs ADC, ABC lie on the same side of AC.
If this be done,
because the angle QOC is equal to the angle POC,
OQ must coincide in direction with OP:
and because OQ is equal to OP,
Q must coincide with P.
Similarly it can be proved that every point on the arc ADC must coincide with some point on the arc ABC,
and every point on the arc ABC with some point on the arc ADC.
Therefore the arc ADC coincides with the arc ABC,
and is equal to it.

Wherefore, *a diameter bisects* &c.

EXERCISES.

1. Prove by superposition that circles, which have equal radii, are equal.

2. Prove by superposition that equal circles have equal radii.

3. Two circles, which have a common centre, but whose radii are not equal, cannot meet.

4. Prove by superposition that two diameters at right angles divide a circle into four equal arcs.

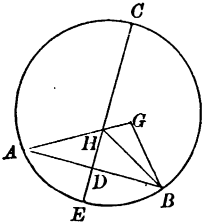
PROPOSITION 2.

If a straight line bisect a chord of a circle at right angles, the line passes through the centre.

Let AB be a chord of the circle ABC, and let CDE be the straight line which bisects AB at right angles:
it is required to prove that CDE passes through the centre of the circle ABC.

CONSTRUCTION. Take any point G not in CE and on the same side of CE as B.

Draw AG cutting CE in H, and draw GB, HB.



PROOF. Because in the triangles ADH, BDH,
AD is equal to BD,
and DH to DH,
and the angle ADH is equal to the angle BDH,
the triangles are equal in all respects; (I. Prop. 4.)
therefore HA is equal to HB.
Therefore GA, which is the sum of GH, HA,
is equal to the sum of GH, HB.
And the sum of GH, HB is greater than GB; (I. Prop. 20.)
therefore GA is greater than GB.
But all straight lines drawn from the centre to a circle are equal. (I. Def. 22.)
Hence the point G cannot be the centre of the circle.
Similarly it can be proved that no point on the same side of CE as A can be the centre;
therefore the centre must be in CE.

Wherefore, *if a straight line* &c.

Corollary 1.

Only one chord drawn through a point within a circle which is not the centre can be bisected at the point.

Corollary 2.

If two chords of a circle bisect each other, their point of intersection is the centre.

EXERCISES.

1. The straight line, which joins the middle points of two parallel chords of a circle, passes through the centre.

2. The locus of the middle points of parallel chords in a circle is a straight line.

3. Two equal parallel chords of a circle are equidistant from the centre.

4. Every parallelogram inscribed in a circle is a rectangle.

5. The diagonals of a rectangle inscribed in a circle are diameters of the circle.

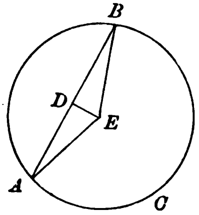
6. If PQ, RS be two parallel chords of a circle and if PR, QS intersect in U and if PS, QR intersect in V, then UV passes through the centre.

PROPOSITION 3.

If a straight line be drawn from the centre of a circle to the middle point of a chord, which is not a diameter, it is at right angles to the chord.

Let *ABC* be a circle and *E* its centre, and let *D* be the middle point of *AB* a chord, which is not a diameter: it is required to prove that *ED* is at right angles to *AB*.

CONSTRUCTION. Draw *EA*, *EB*.



PROOF. Because in the triangles *ADE*, *BDE*,

AD is equal to *BD*,

DE to *DE*,

and *EA* to *EB*,

the triangles are equal in all respects; (I. Prop. 8.)

therefore the angle *ADE* is equal to the angle *BDE*,

and they are adjacent angles.

Therefore the straight lines *ED*, *AB* are at right angles to each other. (I. Def. 11.)

Wherefore, *if a straight line* &c.

EXERCISES.

1. Why are the words "which is not a diameter" inserted in enunciation of Proposition 3?

2. The straight line, which joins the middle points of two parallel chords of a circle, is at right angles to the chords.

3. Circles are described on the sides of a quadrilateral as diameters: shew that the common chord of the circles described on two adjacent sides is parallel to the common chord of the other two circles.

4. A straight line is drawn intersecting two concentric circles: prove that the portions of the straight line which are intercepted between the circles are equal.

5. A straight line cuts two concentric circles in P, R, and Q, S: prove that the rectangle PQ, QR and the rectangle PQ, PS are constant for all positions of the line.

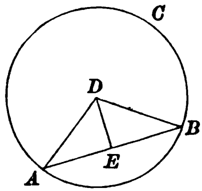
PROPOSITION 4.

If a straight line be drawn from the centre of a circle at right angles to a chord, it bisects the chord.

Let *ABC* be a circle, and *D* its centre, and let the straight line *DE* be drawn at right angles to *AB* a chord, which is not a diameter*:

it is required to prove that *DE* bisects *AB*, that is, that *AE* is equal to *BE*.

CONSTRUCTION. Draw *DA*, *DB*.



PROOF. Because *DB* is equal to *DA*,
each being a radius of the circle,
the angle *DAB* is equal to the angle *DBA*. (I. Prop. 5.)
And the angle *DEA* is equal to the angle *DEB*,
each being a right angle.
Then because in the triangles *EAD*, *EBD*,
the angle *EAD* is equal to the angle *EBD*,
and the angle *DEA* to the angle *DEB*,
and *ED*, a side opposite to a pair of equal angles, is common,
the triangles are equal in all respects;
(I. Prop. 26, Part 2.)
therefore *AE* is equal to *BE*.

Wherefore, *if a straight line* &c.

* The case when the chord is a diameter requires no proof.

In Propositions 2, 3, 4 we have to deal with three properties of a straight line:

(*a*) the passing through the centre of a circle,
(*b*) the being at right angles to a given chord,
(*c*) the bisecting the given chord.

It is proved in these propositions that, if a straight line have any two of these three properties, it necessarily has the third property.

Proposition 2 deduces (*a*) from (*b*) and (*c*);
Proposition 3 deduces (*b*) from (*c*) and (*a*);
Proposition 4 deduces (*c*) from (*a*) and (*b*).

EXERCISES.

1. Two chords are drawn through a point on a circle equally inclined to the radius drawn to the point: prove that the chords are equal.

2. If $ABPQ$, $ABRS$ be two circles and PR, QS be any two parallel straight lines drawn through the points of section, then PR, QS are equal.

3. If A and B be two fixed points and P move so that the perpendicular from A on BP bisects BP, the locus of P is a circle.

4. Draw through a point of intersection of two circles a straight line to make equal chords in the two circles.

5. The locus of the middle points of all chords drawn through a fixed point on a circle is a circle.

6. If two circles PAB, QAB intersect each other at A, the locus of the middle point of a straight line PQ drawn through A is a circle.

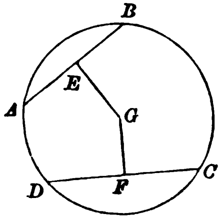
PROPOSITION 5.

To find the centre of a given circle.

Let *ABC* be a given circle:
it is required to find its centre.

Construction. Draw any two chords which are not parallel and which cut the circle in *A*, *B*, and in *C*, *D*.

Bisect *AB* and *CD* at *E* and *F*; (I. Prop. 10.)
and draw *EG*, *FG* at right angles to *AB*, *CD* respectively meeting* at *G*: (I. Prop. 11.)
then *G* is the centre of the circle *ABC*.



Proof. Because the straight line *EG* bisects the chord *AB* at right angles,
EG passes through the centre; (Prop. 2.)
and because the straight line *FG* bisects the chord *CD* at right angles,
FG passes through the centre. (Prop. 2.)
Therefore the point *G*, where the two lines *EG*, *FG* meet, is the centre.

Wherefore, *the centre G of the given circle ABC has been found.*

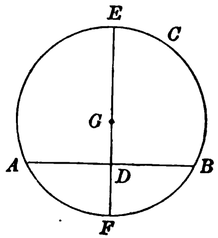
* The lines must meet. See Ex. 2, p. 51.

Outline of Alternative Construction.

Draw any chord AB, of the circle ABC.

Bisect AB in D, and draw EDF at right angles to AB, meeting the circle in E and F.

Bisect EF in G.



It may be proved that

(1) the centre of the circle ABC is in EF;

(2) no other point but G can be the centre.

EXERCISES.

1. Draw all the lines, which are wanted to find the centre of a given circle,

(*a*) in the method given in the text;

(*b*) when the two chords in this method meet on the circle;

(*c*) in the alternative method above.

Which method requires the fewest lines?

2. Draw through a given point within a circle a chord such that it is bisected at the point.

3. Describe a circle with a given centre to cut a given circle at the extremities of a diameter.

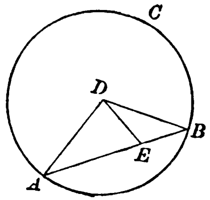
PROPOSITION 6.

Every chord of a circle lies within the circle.

Let AB be the chord joining any two points A, B on the circle ABC:

it is required to prove that any point on the chord AB between A and B is within the circle.

Construction. Find the centre D of the circle; (Prop. 5.) take any point E on AB between A and B and draw DA, DE, DB.



Proof. Because in the triangle DAB, DB is equal to DA,

the angle DAB is equal to the angle DBA; (I. Prop. 5.)

but the exterior angle DEB of the triangle DAE is greater than the interior opposite angle DAE; (I. Prop. 16.)

therefore the angle DEB is greater than the angle DBE.

And because in the triangle DEB, the angle DEB is greater than the angle DBE,

the side DB is greater than the side DE; (I. Prop. 19.)

that is, DE is less than DB which is a radius of the circle.

Therefore the point E is within the circle. (Def. 5.)

But E is *any* point on the chord AB between A and B, and AB is the chord joining *any* two points on the circle.

Wherefore, *every chord of a circle* &c.

EXERCISES.

1. If a chord of a circle be produced, the produced parts lie without the circle.

2. Describe a circle which shall pass through two given points, and which shall have its radius equal to a given straight line greater in length than half the distance between the points.

How many such circles are possible?

3. Draw a straight line to cut two equal circles in P, Q and R, S so that the straight lines PQ, QR, RS may be equal.

What condition is necessary that such a straight line can be drawn?

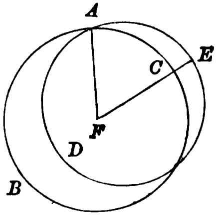
PROPOSITION 7.

If two circles have a common point, they cannot have the same centre.

Let the two circles ABC, ADE meet one another at the point A:
it is required to prove that they cannot have the same centre.

CONSTRUCTION. Find F the centre of one of the circles ABC. (Prop. 5.)

Draw any straight line FCE meeting the circles at two distinct points C and E, and draw FA.



PROOF. Because F is the centre of the circle ABC,
FC is equal to FA. (I. Def. 22.)
But FE is not equal to FC;
therefore FE is not equal to FA;
that is, two straight lines FE, FA drawn to the circle ADE from the point F are not equal.
Therefore F is not the centre of the circle ADE. (I. Def. 22.)

Wherefore, *if two circles* &c.

COROLLARY.

Two concentric circles cannot have a common point.

Definition. *A point is said* **to rotate** *about another point, when the first point moves along a circle, of which the second point is the centre.*

A finite straight line is said **to rotate** *about a point, when each of its extremities moves along a circle, of which the point is the centre, while the line remains of constant length.*

A plane figure is said **to rotate** *about a point, when each of two points fixed in the figure moves along a circle, of which the point is the centre, while the figure remains unchanged in shape and size.*

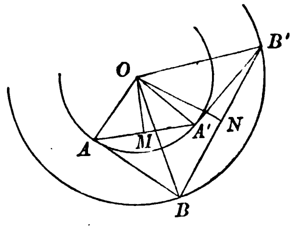
ADDITIONAL PROPOSITION.

Any finite straight line may be shifted from any one position in a plane to any other by rotation about some point in the plane.

Let AB, $A'B'$ be any two positions of a finite straight line in a plane:
it is required to prove that the line can be shifted from the position AB to the position $A'B'$ by rotation about some point in the plane.

Draw AA', BB' and bisect them in M, N, and draw MO, NO at right angles to AA', BB' meeting in O.

Draw OA, OB, OA', OB'.



Because in the triangles AOM, $A'OM$, AM is equal to $A'M$, and OM to OM,
and the angle OMA to the angle OMA',
the triangles are equal in all respects; (I. Prop. 4.)
therefore OA is equal to OA'.

Similarly it can be proved that OB is equal to OB'.

Again, because in the triangles OAB, $OA'B'$,
OA is equal to OA', OB to OB', and AB to $A'B'$,
the triangles are equal in all respects; (I. Prop. 8.)
therefore the angle AOB is equal to the angle $A'OB'$:
add to each the angle BOA';
then the angle AOA' is equal to the angle BOB'.

It appears therefore that the triangle OAB can be shifted into the position $OA'B'$ by being turned in its own plane round the point O through an angle AOA' or BOB';
therefore AB can be shifted to $A'B'$ by rotation round the point O.

EXERCISES.

1. About what point must AB one side of a parallelogram $ABCD$ rotate in order to take (1) the position CD, (2) the position DC?

2. Prove that, when a straight line rotates about a point, every point in the line rotates about the point through the same angle.

3. Any triangle can be shifted from any one position to any other position, which it can occupy in the same plane without being turned over, by rotation about some point in the plane.

4. Prove that, when a plane figure rotates about a point, every point in the figure rotates about the point through the same angle.

5. Describe an equilateral triangle of which one angular point is given and the others lie on two given straight lines.
How many solutions are there?

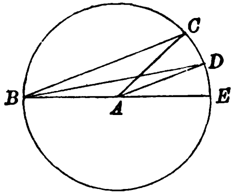
6. Construct an equilateral triangle, one of whose angular points is given and the other two lie one on each of two given circles.
Find the limits of the position of the given point which admit of a possible solution.

7. Construct a square to have one vertex at a fixed point and two opposite vertices on two given straight lines.

PROPOSITION 8. PART 1.

Of all straight lines drawn to a circle from a point on the circle, the line which is a diameter is the greatest; and of any two others, the one which subtends the greater angle at the centre is the greater.

Let CDE be a given circle, A its centre, and B any point on the circle; let BAE be a diameter, and let BC, BD be any other two straight lines drawn from B to the circle, and of the angles BAC, BAD subtended by BC, BD at A let the angle BAD be the greater:
it is required to prove that BE is greater than BD, and BD greater than BC.



PROOF. Because AE is equal to AD,
therefore BE, which is the sum of BA, AE,
is equal to the sum of BA, AD:
but the sum of BA, AD is greater than BD; (I. Prop. 20.)
therefore BE is greater than BD.
Next, because in the triangles BAD, BAC,
AD is equal to AC,
and BA to BA,
and the angle BAD is greater than the angle BAC,
therefore BD is greater than BC. (I. Prop. 24.)
Wherefore, *of all straight lines* &c.

COROLLARY.

A diameter is the greatest chord of a circle.

EXERCISES.

1. If two chords of a circle subtend equal angles at the centre, they are equal.

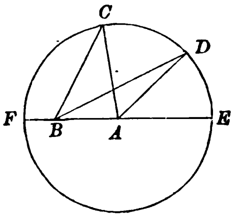
2. If two chords of a circle be equal, they subtend equal angles at the centre.

3. Of any two chords in a circle the one which subtends the greater angle at the centre is the greater.

PROPOSITION 8. PART 2.

Of all straight lines drawn to a circle from an internal point not the centre, the one which passes through the centre is the greatest, and the one which when produced passes through the centre is the least; and of any two others, the one which subtends the greater angle at the centre is the greater.

Let CDE be a given circle, A its centre and B any other internal point; let BA produced beyond A cut the circle in E, and produced beyond B in F, and let BC, BD be any other two straight lines drawn from B to the circle, and of the angles BAC, BAD subtended by BC, BD at A let the angle BAD be the greater:
it is required to prove that BE is greater than BD,
BD greater than BC, and BC greater than BF.



PROOF. Because AE is equal to AD,
therefore BE, which is the sum of BA, AE, is equal to the sum of BA, AD;
but the sum of BA, AD is greater than BD; (I. Prop. 20.)
therefore BE is greater than BD.

Next, because in the triangles BAD, BAC,
AD is equal to AC,
BA to BA,
and the angle BAD is greater than the angle BAC,
therefore BD is greater than BC. (I. Prop. 24.)

Again, because the sum of BC, BA is greater than AC, (I. Prop. 20.)
and AC is equal to AF, which is the sum of BF, BA,
therefore the sum of BC, BA is greater than the sum of BF, BA.
Therefore BC is greater than BF.

Wherefore, *of all straight lines* &c.

EXERCISES.

1. If two straight lines drawn to a circle from an internal point not the centre be equal, they subtend equal angles at the centre.

2. If two straight lines drawn to a circle from an internal point not the centre subtend equal angles at the centre, they are equal.

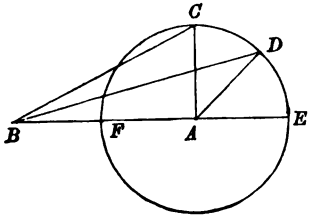
3. If each of two equal straight lines have one extremity on one of two concentric circles and the other extremity on the other, the lines subtend equal angles at the common centre.

PROPOSITION 8. PART 3.

Of all straight lines drawn to a circle from an external point, the one which passes through the centre is the greatest, and the one which when produced passes through the centre is the least; and of any two others, the one which subtends the greater angle at the centre is the greater.

Let *CDE* be a given circle, *A* its centre and *B* a given external point; let *BA* cut the circle in *F* and let *BA* produced cut the circle in *E*, and let *BD*, *BC* be any other two straight lines drawn from *B* to the circle, and of the angles *BAC*, *BAD* subtended by *BC*, *BD* at *A* let the angle *BAD* be the greater:

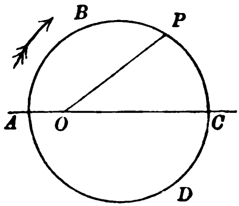
it is required to prove that *BE* is greater than *BD*,
BD greater than *BC*, and *BC* greater than *BF*.



PROOF. Because *AE* is equal to *AD*,
therefore *BE*, which is the sum of *BA*, *AE*,
is equal to the sum of *BA*, *AD*:
but the sum of *BA*, *AD* is greater than *BD*; (I. Prop. 20.)
therefore *BE* is greater than *BD*.
Next, because in the triangles *BAD*, *BAC*,
AD is equal to *AC*,
and *BA* to *BA*,
and the angle *BAD* is greater than the angle *BAC*,
therefore *BD* is greater than *BC*. (I. Prop. 24.)
Again, because the sum of *BC*, *CA* is greater than *BA*, (I. Prop. 20.)
which is the sum of *BF*, *FA*,
and because *CA* is equal to *FA*,
therefore *BC* is greater than *BF*.

Wherefore, *of all straight lines* &c.

We conclude from the results of the several Parts of Proposition 8 that, if O be a fixed point on the diameter AC of a circle $ABCD$ nearer to A than to C and P be a point which is capable of motion along the circumference in the direction represented by the arrow, while P is moving along the arc ABC from A to C the distance OP increases continuously from OA to OC, and while P is moving along the arc CDA from C to A, the distance OP decreases continuously from OC to OA.



We say therefore that OC is a *maximum* value of OP, and OA is a *minimum* value. (See remarks on page 55.)

It may be observed here that, if P travel round the circle any number of times, it passes C and A alternately. It appears therefore that here *maximum* and *minimum* values occur alternately.

The occurrence of *maximum* and *minimum* values alternately is true generally in the case of quantities which vary continuously, i.e. quantities whose magnitude changes without suffering any abrupt changes.

EXERCISES.

1. Find the shortest distance between two points one on each of two circles which do not meet.

2. A and B are two fixed points; it is required to find a point P on a given circle, so that the sum of the squares on AP and BP may be the least possible.

Under what conditions is the solution indeterminate?

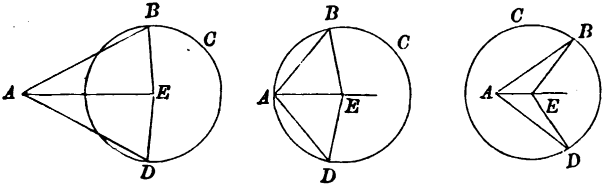
PROPOSITION 9.

From a point not the centre not more than two equal straight lines can be drawn to a circle, one on each side of the straight line drawn from that point to the centre.

Let A be a given point, and BCD a given circle,
and let AB, AD be two equal straight lines drawn from A to the circle:
it is required to prove that no other straight line equal to AB or AD can be drawn from A to the circle.

CONSTRUCTION. Find E the centre of the circle; (Prop. 5.)

draw EA, EB, ED.



PROOF. Take C any point of the circle on the same side of AE as AB.

Because B and C are equidistant from E,
they cannot be equidistant from A. (I. Prop. 7.)
Therefore there cannot be another straight line equal to AB drawn from A to the circle on the same side of AE as AB.
Similarly it can be proved that there cannot be another straight line equal to AD drawn from A to the circle on the same side of AE as AD.

Wherefore, *from a point* &c.

COROLLARY 1.

If from a point three equal straight lines can be drawn to a circle, that point is the centre.

COROLLARY 2.

Two circles cannot meet in more than two points.

If the straight line AB drawn to the circle from a point A not the centre be in the same straight line as the centre of the circle, no other straight line can be drawn to the circle from the point A equal to AB. The line AB is in this case either a *maximum* or a *minimum* among the straight lines drawn from the point A to the circle; it is a *maximum*, if B be at the further extremity of the diameter through A, and a *minimum*, if B be at the nearer extremity.

EXERCISES.

1. If from any point within a circle two straight lines be drawn to the circumference making equal angles with the straight line joining the point and the centre, the lines are equal in length.

2. Construct an equilateral triangle, having two of its vertices on a given circle and the third at a given point within the circle.

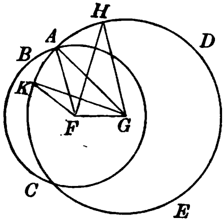
3. Construct a square having one vertex at a given point and two opposite vertices on a given circle.

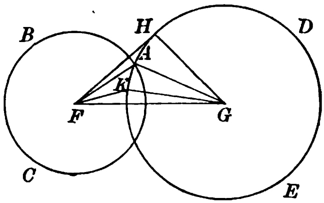
PROPOSITION 10.

If two circles meet at a point not in the same straight line as their centres, the circles intersect at that point.

Let *ABC*, *ADE* be two circles meeting at a point *A*, which is not in the same straight line as their centres: it is required to prove that the circles intersect at *A*.

CONSTRUCTION. Find *F*, *G* the centres of the circles *ABC*, *ADE*; (Prop. 5.)
draw *AF*, *FG*, *GA*; and through *G* draw, on the same side of *FG* as *GA*, two straight lines *GH*, *GK* to meet the circle *ADE* in *H*, *K*, such that the angle *FGH* is greater than the angle *FGA*, and the angle *FGK* less than the angle *FGA*.
Draw *FH*, *FK*.





PROOF. Because, from the point *F* not the centre of the circle *ADE* the straight lines *FH*, *FA*, *FK* are drawn to the circle,
such that the angle *FGH* subtended by *FH* at *G* the centre is greater than the angle *FGA* subtended by *FA*,
and such that the angle *FGA* is greater than the angle *FGK* subtended by *FK*,

FH is greater than *FA*,
and *FA* greater than *FK*. (Prop. 8, Parts 2 and 3.)
But *FA* is a radius of the circle *ABC*:

therefore the distance of the point H from the centre of the circle ABC is greater than the radius, and H therefore is without the circle ABC;
and the distance of the point K from the centre of the circle ABC is less than the radius, and K therefore is within the circle ABC.
Therefore the circles intersect at the point A. (Def. 5.)

Wherefore, *if two circles* &c.

COROLLARY.

If two circles touch, the point of contact is in the same straight line as their centres.

EXERCISES.

1. If two circles meet at a point not in the same straight line as their centres, the circles meet at one other point.

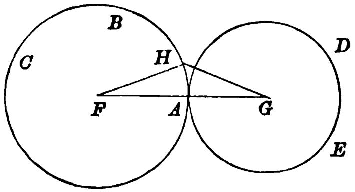
2. The straight line joining the two points at which two circles meet is bisected at right angles by the straight line joining the centres.

PROPOSITION 11.

If two circles meet at a point, which lies in the same straight line as their centres and is between the centres, the circles touch at that point, and each circle lies without the other.

Let ABC, ADE be two circles meeting at a point A, which is in the same straight line as their centres, and is between the centres:
it is required to prove that the circles touch at A, and that each circle lies without the other.

CONSTRUCTION. Find the centres F, G of the circles ABC, ADE; (Prop. 5.)
draw FG, which by the hypothesis passes through A.
Take any point H on the circle ABC,
and draw FH, HG.

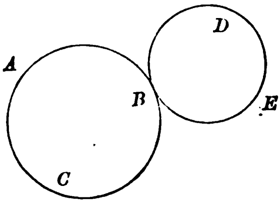


PROOF. Because the sum of FH, HG is greater than FG, (I. Prop. 20.)
that is, greater than the sum of FA, AG,
and FH is equal to FA,
therefore HG is greater than AG.
But AG is a radius of the circle ADE;
therefore the distance of the point H from the centre of the circle ADE is greater than the radius, and H therefore is without the circle ADE. (Def. 5.)
Therefore every point on the circle ABC except A lies without the circle ADE.
Therefore the circles touch at A. (Def. 7.)
Similarly it can be proved that every point on the circle ADE except A lies without the circle ABC.

Wherefore, *if two circles* &c.

When one circle touches another circle which lies without it, the first circle is said to have **external contact** with the second circle.

In the diagram each of the circles *ABC*, *BDE* has external contact with the other at the point *B*.



EXERCISES.

1. If the distance between the centres of two circles be greater than the sum of their radii, each circle lies without the other.

2. Prove that in all cases the greatest distance between two points one on each of two given circles is greater than the distance between the centres by the sum of the radii.

3. Of two equal circles of given radius, which touch externally at *P*, one touches *OX* and the other touches *OY*, where *OX*, *OY* are two given straight lines at right angles to each other: prove that the locus of *P* is an equal circle.

Shew that there are four such loci.

4. If two equal circles touch, every straight line drawn through the point of contact will make equal chords in the two circles.

5. Given two concentric circles, draw a chord of the outer which shall be trisected by the inner circle.

6. Three circles touch one another externally at the points *A*, *B*, *C*; the straight lines *AB*, *AC* are produced to cut the circle *BC* at *D* and *E*: shew that *DE* is a diameter of *BC*, and is parallel to the straight line joining the centres of the other circles.

PROPOSITION 12.

If two circles meet at a point, which lies in the same straight line as their centres and is not between the centres, the circles touch at that point, and one of the circles lies within the other.

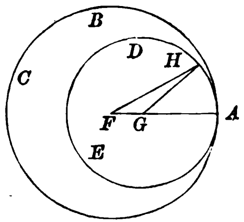
Let ABC, ADE be two circles meeting at a point A, which is in the same straight line as their centres and is not between the centres:

it is required to prove that the circles touch at A, and that one of the circles lies within the other.

CONSTRUCTION. Find the centres F, G of the circles ABC, ADE; (Prop. 5.)

draw FG, and produce FG which by the hypothesis passes through A. Let ADE be the circle whose centre G is the nearer to A.

Take H any point on the circle ADE, and draw FH, HG.



PROOF. Because GA is equal to GH,

FA, which is the sum of FG, GA, is equal to the sum of FG, GH;

but the sum of FG, GH is greater than FH; (I. Prop. 20.)

therefore FA is greater than FH.

But FA is a radius of the circle ABC;

therefore the distance of the point H from the centre of the circle ABC is less than the radius, and H therefore is within the circle ABC;

therefore every point on the circle ADE except A lies within the circle ABC.

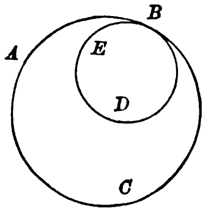
Therefore the circles touch at A. (Def. 7.)

Wherefore, *if two circles* &c.

When one circle touches another circle, which lies within it, the first circle is said to have **internal contact** with the second circle.

Two circles can have external contact with each other, but two circles cannot have internal contact with each other. If one circle have *internal* contact with another circle, the second circle has *external* contact with the first circle.

In the diagram the circle *ABC* has **internal** contact with the circle *BDE*, but the circle *BDE* has **external** contact with the circle *ABC* at the point *B*.



EXERCISES.

1. Describe a circle passing through a given point and touching a given circle at a given point.

2. If in any two given circles which touch one another, there be drawn two parallel diameters, the point of contact and an extremity of each diameter, lie in the same straight line.

3. Describe a circle which shall touch a given circle, have its centre in a given straight line, and pass through a given point in the given straight line.

4. Describe a circle of given radius to pass through a given point and to touch a given circle.

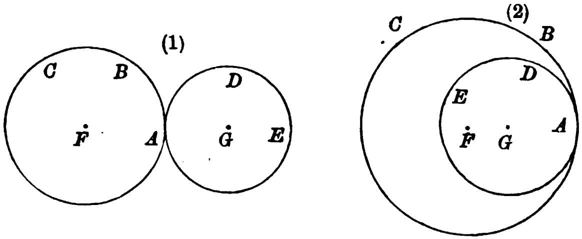
What conditions are necessary that a solution may be possible?

PROPOSITION 13.

If two circles have a point of contact, they do not meet at any other point.

Let *ABC*, *ADE* be two circles which touch at the point *A*:
it is required to prove that the circles do not meet at any other point.

CONSTRUCTION. Find *F*, *G* the centres of the circles *ABC*, *ADE*. (Prop. 5.)



PROOF. Because the circles *ABC*, *ADE* touch,
the point of contact *A* must lie in the straight line *FG* or in *FG* produced. (Prop. 10, Coroll.)

First (fig. 1) let the point *A* lie in *FG*:
then because the circles *ABC*, *ADE* meet at a point *A* in the same straight line *FG* as their centres and between the centres,

each circle lies without the other. (Prop. 11.)

Secondly (fig. 2) let the point *A* lie in *FG* produced:
then because the circles *ABC*, *ADE* meet at a point *A* in the same straight line *FG* as their centres and not between the centres,

one circle lies within the other. (Prop. 12.)

Therefore in neither case can the circles meet at any point other than *A*.

Wherefore, *if two circles* &c.

We infer as a result of Propositions 9—13 that two circles must be such that they either

(*a*) intersect in two distinct points,

or (*b*) touch at one point, which is in the straight line joining the centres,

or (*c*) do not meet.

EXERCISES.

1. What is the greatest number of contacts which may exist among (1) three, (2) four circles?

2. Describe three circles to have their centres at three given points, and to touch each other in pairs.

3. Into how many parts will three circles divide a plane? Distinguish between the different cases which may occur, when the circles intersect or touch.

PROPOSITION 14. PART 1.

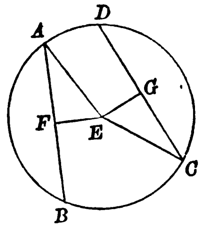
Chords of a circle, which are equal, are equidistant from the centre.

Let AB, CD be two equal chords of the circle $ABCD$: it is required to prove that AB, CD are equidistant from the centre.

CONSTRUCTION. Find E the centre of the circle $ABCD$; (Prop. 5.)

and from E draw EF, EG at right angles to AB, CD. (I. Prop. 12.)

Draw EA, EC.



PROOF. Because the straight line EF is drawn through the centre of the circle at right angles to the chord AB, it bisects it; (Prop. 4.)

that is, AB is double of AF.

Similarly it can be proved that CD is double of CG.

But AB is equal to CD;

therefore AF is equal to CG.

Next, because the angles AFE, CGE are right angles, the square on AE is equal to the sum of the squares on AF, FE,

and the square on CE is equal to the sum of the squares on CG, GE. (I. Prop. 47.)

And because AE is equal to CE,

the square on AE is equal to the square on CE.

Therefore the sum of the squares on AF, FE is equal to the sum of the squares on CG, GE.

Because AF is equal to CG,

the square on AF is equal to the square on CG;

therefore the square on FE is equal to the square on GE.

Therefore FE is equal to GE,
that is, AB, CD are distant from the centre.
(Def. 9.)

Wherefore, *chords of a circle* &c.

EXERCISES.

1. Chords of a circle, which are equal, subtend equal angles at the centre.

2. Chords of a circle, which subtend equal angles at the centre, are equidistant from the centre.

PROPOSITION 14. PART 2.

Chords of a circle, which are equidistant from the centre, are equal.

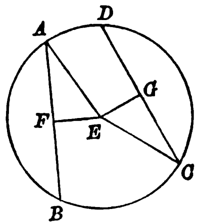
Let AB, CD be two chords of the circle $ABCD$ equidistant from the centre:

it is required to prove that AB is equal to CD.

CONSTRUCTION. Find E the centre of the circle $ABCD$; (Prop. 5.)

and from E draw EF, EG at right angles to AB, CD. (I. Prop. 12.)

Draw EA, EC.



PROOF. Because the straight line EF is drawn through the centre of the circle at right angles to the chord AB, it bisects it; (Prop. 4.)

that is, AB is double of AF.

Similarly it may be proved that CD is double of CG.

Next, because the angles AFE, CGE are right angles, the square on AE is equal to the sum of the squares on AF, FE,

and the square on CE is equal to the sum of the squares on CG, GE. (I. Prop. 47.)

And because AE is equal to CE,

the square on AE is equal to the square on CE.

Therefore the sum of the squares on AF, FE is equal to the sum of the squares on CG, GE.

Because EF is equal to EG,

the square on EF is equal to the square on EG;

therefore the square on AF is equal to the square on CG;

therefore AF is equal to CG.

And it has been proved that AB is double of AF, and CD of CG.

Therefore AB is equal to CD.

Wherefore, *chords of a circle* &c.

Parts 1 and 2 of Proposition 14 are the converses of each other.

EXERCISES.

1. In a circle chords, which are equidistant from the centre, subtend equal angles at the centre.

2. In a circle chords, which subtend equal angles at the centre, are equal.

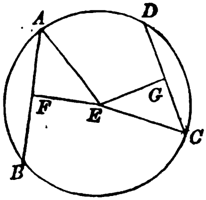
PROPOSITION 15. PART 1.

Of any two chords of a circle the one which is the greater is the nearer to the centre.

Let *AB*, *CD* be two chords of the circle *ABCD*, of which *AB* is greater than *CD*:
it is required to prove that *AB* is nearer to the centre than *CD*.

CONSTRUCTION. Find *E* the centre of the circle *ABCD*; (Prop. 5.)
and from *E* draw *EF*, *EG* at right angles to *AB*, *CD*. (I. Prop. 12.)

Draw *EA*, *EC*.



PROOF. Because the straight line *EF* is drawn through the centre at right angles to the chord *AB*,

AF is equal to *FB*, (Prop. 4.)

and *AB* is double of *AF*.

Similarly it can be proved that *CD* is double of *CG*.

But *AB* is greater than *CD*;

therefore *AF* is greater than *CG*.

Next, because the angles *AFE*, *CGE* are right angles, the square on *AE* is equal to the sum of the squares on *AF*, *FE*,
and the square on *CE* is equal to the sum of the squares on *CG*, *GE*. (I. Prop. 47.)

And because *AE* is equal to *CE*,

the square on *AE* is equal to the square on *CE*.

Therefore the sum of the squares on *AF*, *FE* is equal to the sum of the squares on *CG*, *GE*;

Because AF is greater than CG,
the square on AF is greater than the square on CG;
therefore the square on FE is less than the square on GE.
Therefore FE is less than GE,
that is, AB is nearer to the centre than CD.

Wherefore, *of any two chords* &c.

EXERCISES.

1. Prove that every straight line, which makes equal chords in two equal circles, is parallel to the straight line joining the centres or passes through the middle point of that line.

2. Find the shortest chord which can be drawn through a given point within a circle.

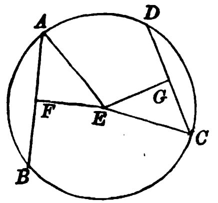
PROPOSITION 15. PART 2.

Of any two chords of a circle the one which is the nearer to the centre is the greater.

Let AB, CD be two chords of the circle $ABCD$, of which AB is nearer to the centre than CD:
it is required to prove that AB is greater than CD.

CONSTRUCTION. Find E the centre of the circle $ABCD$; (Prop. 5.)
and from E draw EF, EG at right angles to AB, CD. (I. Prop. 12.)

Draw EA, EC.



PROOF. Because the straight line EF is drawn through the centre at right angles to the chord AB,

AF is equal to FB, (Prop. 4.)
and AB is double of AF.

Similarly it can be proved that CD is double of CG.

Next, because the angles AFE, CGE are right angles, the square on AE is equal to the sum of the squares on AF, FE,
and the square on CE is equal to the sum of the squares on CG, GE. (I. Prop. 47.)

And because AE is equal to CE,
the square on AE is equal to the square on CE.

Therefore the sum of the squares on AF, FE is equal to the sum of the squares on CG, GE.

Because EF is less than EG,
the square on EF is less than the square on EG;

therefore the square on AF is greater than the square on CG;

therefore AF is greater than CG.

Therefore AB is greater than CD.

Wherefore, *of any two chords* &c.

Parts 1 and 2 of Proposition 15 are the converses of each other.

EXERCISES.

1. Of any two chords of a circle the nearer to the centre subtends the greater angle at the centre.

2. Draw through a given point a straight line to make equal chords in two given equal circles.

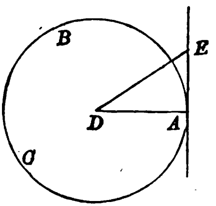
Discuss the number of possible solutions in the different cases which may occur.

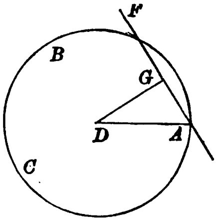
PROPOSITION 16.

The straight line drawn through a point on a circle at right angles to the radius touches the circle, and every other straight line drawn through the point cuts the circle.

Let ABC be a circle, of which D is the centre, and let AE be a straight line drawn through A at right angles to the radius AD; and let AF be any other straight line drawn through A:
it is required to prove that AE touches the circle, and that AF cuts the circle.

Construction. Take any point E on AE, and draw DE, and from D draw DG at right angles to AF.
(I. Prop. 12.)





Proof. Because in the triangle DAE,
the angle DAE is a right angle,
the angle DEA is less than a right angle; (I. Prop. 17.)
therefore the angle DAE is greater than the angle DEA;
therefore DE is greater than DA. (I. Prop. 19.)

Therefore the distance of the point E from the centre is greater than the radius, and E therefore is without the circle. (Def. 5.)

Similarly it can be proved that every point on AE except A is without the circle.

Therefore AE touches the circle ABC at A. (Def. 6.)

Next because in the triangle DGA,
the angle DGA is a right angle,
the angle DAG is less than a right angle; (I. Prop. 17.)
therefore the angle DAG is less than the angle DGA;
therefore DG is less than DA. (I. Prop. 19.)

Therefore the distance of the point G from the centre is less than the radius, and therefore G is within the circle. (Def. 5.)

Therefore the straight line AF cuts the circle ABC *.

Wherefore, *the straight line* &c.

We infer as a result of Proposition 16 that a straight line and a circle must be such that they either

(*a*) intersect in two distinct points,
or (*b*) touch at one point,
or (*c*) do not meet.

EXERCISES.

1. A point B is taken on a circle whose centre is C; PA a tangent at any point P meets CB produced at A, and PD is drawn perpendicular to CB: prove that PB bisects the angle APD.

2. Describe a circle to have its centre on a given straight line, to pass through a given point on that line and to touch another given straight line.

3. Describe a circle to pass through a given point and to touch a given straight line at a given point.

4. If AC be a diameter of a circle ABC, and AP be drawn perpendicular to the tangent at B, AB bisects the angle CAP.

5. Prove that although no straight line can be drawn to pass between a circle and its tangent, yet any number of circles can be described to do so.

6. Circles, which have a common tangent at a point, touch each other.

7. Prove that the angle between a tangent to a circle and a chord drawn from the point of contact is half of the angle subtended at the centre by the chord.

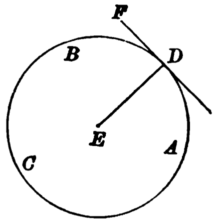
* A straight line which cuts a circle is often called a secant.

PROPOSITION 17.

Through a given point to draw a tangent to a given circle.

Let ABC be a given circle, and D a given point: it is required through D to draw a tangent to the circle ABC.

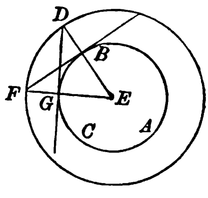
First, let the point D be on the circle.



CONSTRUCTION. Find the centre E; (Prop. 5.)
draw ED, and draw DF at right angles to DE; (I. Prop. 11.)
then DF is a tangent drawn as required.

PROOF. Because the straight line DF is drawn through the point D on the circle ABC at right angles to DE the radius, DF touches the circle. (Prop. 16.)

Secondly, let the point D be outside the circle.



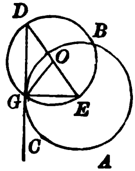
CONSTRUCTION. Find the centre E; (Prop. 5.)
draw ED, cutting the circle ABC between E and D in B, and draw BF at right angles to EB, (I. Prop. 11.) and with E as centre and ED as radius describe a circle cutting BF in F. Draw EF, cutting the circle ABC between E and F in G, and draw DG:
then DG is a tangent drawn as required.

PROOF. Because in the triangles DEG, FEB,
DE is equal to FE, EG to EB, and the angle DEG equal to the angle FEB,
the triangles are equal in all respects; (I. Prop. 4.)
therefore the angle DGE is equal to the angle FBE.
But the angle FBE is a right angle;
therefore the angle DGE is a right angle;

and because the straight line GD is drawn through the point G on the circle ABC at right angles to GE the radius, GD touches the circle. (Prop. 16.)

Wherefore, *through a given point D a tangent has been drawn to the circle ABC.*

Outline of Alternative Construction.



Find E the centre. (Prop. 5.)

Draw ED, and bisect it in O. (I. Prop. 10.)

With O as centre and OD as radius describe a circle, cutting the circle ABC in G.

Draw DG, OG, EG.

It may be proved that

(1) the angle OGD is equal to the angle GDO,

(2) the angle OGE is equal to the angle GED,

and (3) the angle EGD is a right angle,

and hence that GD is a tangent to the circle ABC at G.

Both the construction in the Proposition and the alternative construction point out that two and only two tangents can be drawn to a circle through an external point, one through a point on the circle, and none through an internal point.

When there is no danger of ambiguity the length of the straight line drawn from an external point to touch a circle which is intercepted between that point and the point of contact is often spoken of as the tangent from the point to the circle.

EXERCISES.

1. The two tangents drawn to a circle from an external point are equal.

2. Draw a tangent to a given circle to be parallel to a given straight line.

3. Find in a given straight line a point such that the tangent drawn from it to a given circle may be equal to a given straight line.

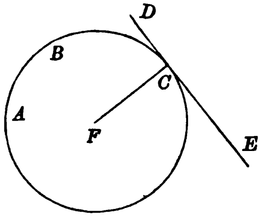
4. The greater the distance of an external point is from the centre of a circle, the smaller is the inclination of the two tangents which can be drawn from it.

PROPOSITION 18.

If a straight line touch a circle, the radius drawn to the point of contact is at right angles to the line.

Let the straight line DE touch the circle ABC at the point C:
it is required to prove that the radius drawn to the point C is at right angles to DCE.

CONSTRUCTION. Find F the centre of the circle; (Prop. 5.)
and draw FC.

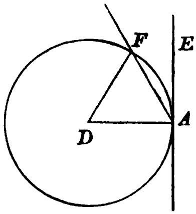


PROOF. If DCE were not at right angles to CF,
DCE would cut the circle (Prop. 16);
but it does not:
therefore DCE is at right angles to CF.

Wherefore, *if a straight line &c.*

The Tangent as the Limit of the Secant.

Let *AF* be a straight line cutting a given circle at a given point *A*, and again at a second point *F*.



Let *D* be the centre of the circle, and *AE* the tangent at *A*. Draw *DA*, *DF*.

The angle *FAE* is equal to half of the angle *ADF*. (See Ex. 7, p. 213.)

Hence the smaller the angle *ADF* is, or the smaller the chord *AF*, (Prop. 8, Part 1.)

the smaller is the angle *FAE*.

Now because we can take the point *F* as close to *A* as we like, we can make the angle *ADF*, and therefore also the angle *FAE*, as small as we like. Hence we can make the straight line *AF* deviate as little as we please from coincidence with *AE*.

We express this fact by saying that,

the tangent AE is the limit of the secant AF, when F moves up close to A.

This definition of a tangent to a curve as the limit of the secant through the point is one which *admits of application to curves of all kinds*.

EXERCISES.

1. Through a given point draw a straight line so that the chord which is intercepted on it by a given circle is equal to a given straight line.

2. Two circles are concentric: prove that all chords of the outer circle which touch the inner are equal.

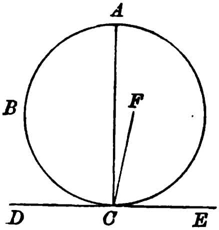
3. If two tangents be drawn to a circle from an external point, the chord joining the points of contact is bisected at right angles by the straight line joining the centre and the external point.

PROPOSITION 19.

If a straight line touch a circle, the straight line drawn at right angles to the line through its point of contact passes through the centre.

Let the straight line DE touch the circle ABC at C, and from C let CA be drawn at right angles to DE: it is required to prove that the centre of the circle is in CA.

CONSTRUCTION. Take any point F, not in CA, and draw FC.



PROOF. Because CA is at right angles to DE,
CF cannot be at right angles to DE. (I. Prop. 10 A.)

But if a straight line touch a circle, the radius drawn to the point of contact is at right angles to the tangent; (Prop. 18.)

therefore the radius drawn to C cannot be in the same straight line as CF;

therefore the centre cannot lie at any point F not in CA, that is, the centre must lie in CA.

Wherefore, *if a straight line* &c.

ADDITIONAL PROPOSITION.

To draw a common tangent to two given circles.

Let A, B be the centres of two given circles, which we will call for shortness the (A) circle and the (B) circle :

Let the radius of the (A) circle be greater than the radius of the (B) circle.

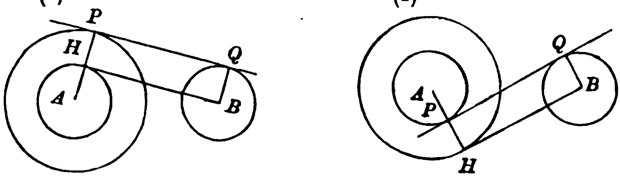
With A for centre and the difference (fig. 1) or the sum (fig. 2) of the radii for radius describe a circle,

and from B draw BH a tangent to it. (Prop. 17.)

Draw AH and let AH produced (fig. 1) or AH (fig. 2) cut the (A) circle in P.

Through P, B draw PQ, BQ parallel to HB, AH respectively. (I. Prop. 31.)

(1) (2)



Because $HPQB$ is a parallelogram, (Constr.)
BQ is equal to HP, (I. Prop. 34.)
which is equal to the radius of the (B) circle. (Constr.)
Therefore the point Q is on the (B) circle.

Again, because BH is a tangent at H,
the angle PHB is a right angle; (Prop. 18.)
therefore the parallelogram $HPQB$ is a rectangle; (I. Def. 19.)
therefore the angles at P and Q are right angles, (I. Prop. 29, Coroll.)
and PQ is a tangent to the (A) and (B) circles at P and Q respectively. (Prop. 16.)

EXERCISES.

1. Prove that four common tangents can be drawn to two circles which are external to each other.

2. How many common tangents can be drawn to two intersecting circles?

3. Is it possible that two circles can have one and only one common tangent?

4. Draw a straight line so that the chords which are intercepted on it by two given circles are equal to two given straight lines.

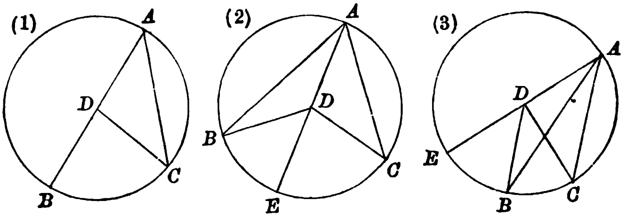
PROPOSITION 20.

The angle which an arc of a circle subtends at the centre is double of the angle which the arc subtends at the circumference.

Let *ABC* be a circle, of which *BC* is an arc, and let *BDC*, *BAC* be angles subtended by the arc *BC* at the centre *D*, and at the circumference:
it is required to prove that the angle *BDC* is double of the angle *BAC*.

First, (fig. 1) let the centre *D* lie on *AB*, one of the lines which contain the angle *BAC*.

CONSTRUCTION. Draw *DC*.



PROOF. Because *DA* is equal to *DC*,
the angle *DCA* is equal to the angle *DAC*; (I. Prop. 5.)
therefore the sum of the angles *DAC*, *DCA* is double of the angle *DAC*.

But the angle *BDC* is equal to the sum of the angles *DAC*, *DCA*; (I. Prop. 32.)
therefore the angle *BDC* is double of the angle *DAC*.

Next, let the centre *D* lie within (fig. 2) or without (fig. 3) the angle *BAC*.

CONSTRUCTION. Draw *AD* and produce it to meet the circle in *E*.

PROOF. It follows from the first case, that the angle *EDC* is double of the angle *EAC*, and that the angle *EDB* is double of the angle *EAB*;

therefore in (fig. 2) the sum of the angles *EDC*, *EDB* is double of the sum of the angles *EAC*, *EAB*,
and in (fig. 3) the difference of the angles *EDC*, *EDB* is double of the difference of the angles *EAC*, *EAB*;
therefore in all cases the angle *BDC* is double of the angle *BAC*.

Wherefore, *the angle which an arc* &c.

In the diagram of Proposition 20 in each of the figures the angle *BDC* is double of the angle *BAC*. Now it is easily seen that although in figures (1) and (3) the angle *BAC* is restricted to values less than a right angle, and the angle *BDC* in consequence to values less than two right angles, in figure (2) the angle *BAC* is restricted only to values less than two right angles and the angle *BDC* in consequence only to values less than four right angles. It appears therefore that, if we wish not to destroy the generality of the theorem of Proposition 20, we must allow our definition of an angle to include angles which are equal to two or greater than two right angles; there is nothing inconsistent with a strict adherence to Euclid's methods in doing so.

EXERCISES.

1. Two circles, whose centres are *A* and *D*, touch externally at *E*: a third circle, whose centre is *B*, touches them internally at *C* and *F*: prove that the angle *ADB* is double of the angle *ECF*.

2. If *AB* be a fixed diameter and *DE* an arc of constant length in a fixed circle, and the straight lines *AE*, *BD* intersect at *P*, the angle *APB* is constant.

3. If *ABC* be a triangle inscribed in a circle and the angle *BAC* be bisected by *AD*, which meets the circle in *D*, then the diameter through *D* will bisect *BC* at right angles.

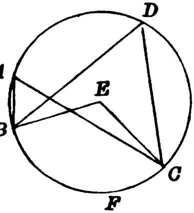
4. *AB* is a diameter and *PQ* any chord of a circle cutting *AB* within the circle, and *AL* is drawn perpendicular to *PQ*. Prove that the angle *LAB* is equal to the sum of the angles *PAB*, *QAB*.

PROPOSITION 21.

Angles in the same arc of a circle are equal.

Let $ABCD$ be a circle, and BAC, BDC be two angles in the same arc $BADC$:
it is required to prove that the angles BAC, BDC are equal.

CONSTRUCTION. Find the centre E; (Prop. 5.)
and draw EB, EC.



PROOF. Because the angle which the arc BFC of the circle subtends at the centre is double of the angle which it subtends at the circumference,

the angle BEC is double of the angle BAC,
and also the angle BEC is double of the angle BDC; (Prop. 20.)
therefore the angle BAC is equal to the angle BDC.

Wherefore, *angles in the same arc* &c.

COROLLARY. *If a straight line joining two points subtend equal angles at two other points on the same side of the line, the four points lie on a circle.*

Let the straight line BC subtend equal angles at the two points A, D on the same side of BC.

If a circle be described about the triangle BAC*, the

* That it is possible to describe a circle through the three vertices of a triangle appears in the Additional Proposition on page 53.

circle must cut BD again at some point not on the same side of AC as B. (Prop. 6.)

Now take H any point but D in BD or BD produced and draw HD.

Then the angle BHC cannot be equal to the angle BDC, (I. Prop. 16.)

and therefore cannot be equal to the angle BAC.

But angles in the same arc of a circle are equal. (Prop. 21.)

Therefore the circle BAC cannot meet BD in H;

that is, it must meet it in D.

In some books in the proof of Proposition 21, the result of Proposition 20 is quoted as if it were true only in the case of arcs greater than a semicircle: that is, as if the angle BEC, which the arc BFC subtends at the centre, were restricted to magnitudes less than two right angles. The general truth of the theorem is then deduced as a consequence.

We leave this deduction to the student as an exercise.

EXERCISES.

1. The locus of a point at which a given straight line subtends a constant angle is an arc of a circle.

2. If of three concurrent straight lines inclined at given angles to one another two pass through two fixed points, the third also passes through a third fixed point.

3. If two sides of a triangle of constant shape and size pass through two fixed points, the third always touches a fixed circle.

4. If two sides of a triangle of constant shape and size always touch two fixed circles, the third side always touches a fixed circle.

5. If ABC be an equilateral triangle described in a circle whose centre is O, and if AO produced meet the circle in D, then OD, BC bisect each other.

6. Two circles ADB, ACB intersect in points A and B. Through A any chord DAC is drawn, and BC, BD are joined, and the angle DBC is internally bisected by a line BE which meets DC in E. Shew that E lies on a fixed circle.

7. If ABC be an isosceles triangle on the base BC, inscribed in a circle, and P, Q be points on the arcs AC, AB respectively of the circle such that AQ is parallel to BP, then CQ is parallel to AP.

8. If the diagonals of a quadrilateral inscribed in a circle be at right angles, the perpendicular from their intersection on any side bisects the opposite side.

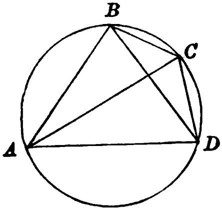
PROPOSITION 22.

The sum of two opposite angles of a convex quadrilateral inscribed in a circle is equal to two right angles.

Let $ABCD$ be a quadrilateral inscribed in the circle $ABCD$:

it is required to prove that the sum of the angles ABC, ADC is equal to two right angles, and that the sum of the angles BAD, BCD is equal to two right angles.

CONSTRUCTION. Draw AC, BD.



PROOF. Because the angles BCA, BDA are in the same arc $BCDA$,

the angle BCA is equal to the angle BDA; (Prop. 21.)

and because the angles CAB, CDB are in the same arc $CDAB$,

the angle CAB is equal to the angle CDB. (Prop. 21.)

Therefore the sum of the angles BCA, CAB is equal to the sum of the angles BDA, CDB, that is, to the angle ADC.

To each of these equals add the angle ABC:

then the sum of the angles ABC, BCA, CAB is equal to the sum of the angles ABC, ADC.

But because the angles ABC, BCA, CAB are the angles of a triangle, their sum is equal to two right angles. (I. Prop. 32.)

Therefore the sum of the angles ABC, ADC is equal to two right angles.

Similarly it can be proved that the sum of the angles BAD, BCD is equal to two right angles.

Wherefore, *the sum of two opposite angles* &c.

Corollary. *If the sum of two opposite angles of a convex quadrilateral be equal to two right angles, the vertices of the quadrilateral lie on a circle.*

Let $ABCD$ be a convex quadrilateral in which the sum of the angles BAD, BCD is equal to two right angles.

If a circle be described about the triangle BAD, the circle must cut AC again in some point not on the same side of BD as A. (Prop. 6.)

Now take H any point but C in AC or AC produced and draw HB, HD.

Then the angle BHD cannot be equal to the angle BCD. (I. Prop. 21.)

Therefore the sum of the angles BAD, BHD cannot be equal to two right angles.

But the sum of two opposite angles of a convex quadrilateral inscribed in a circle is equal to two right angles. (Prop. 22.)

Therefore the circle BAD cannot meet AC in H;
that is, it must meet it in C.

EXERCISES.

1. If the sides AB, DC of a quadrilateral $ABCD$ inscribed in a circle be produced to meet at E, the triangles AEC, BED are equiangular to one another.

2. A triangle is inscribed in a circle: shew that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

3. If $PQRS$, $pqrs$ be two circles, and $PprR$, $QqsS$ be chords such that P, p, q, Q lie on a circle, then R, r, s, S lie on a circle.

4. If any two consecutive sides of a convex hexagon inscribed in a circle be respectively parallel to their opposite sides, the remaining sides are parallel to each other.

5. If any arc of a circle described on the side BC of a triangle ABC cut BA, CA produced if necessary in P and Q, PQ is always parallel to a fixed straight line.

6. E is a point on one of the diagonals AC of a parallelogram $ABCD$. Circles are described about DEA and BEC. Shew that BD passes through the other point of intersection of the circles.

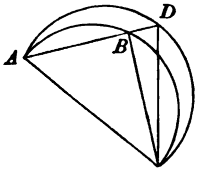
PROPOSITION 23.

Two arcs of circles, which have a common chord and are on the same side of it, cannot be similar unless they are coincident.

Let *ABC*, *ADC* be two arcs of circles, which have a common chord *AC*, and are on the same side of it:

it is required to prove that *ABC*, *ADC* cannot be similar arcs, unless they are coincident.

Construction. Draw through *A*, one of the extremities of the chord *AB*, any straight line *ABD* to meet the arcs in *B*, *D*; and draw *CB*, *CD*.



Proof. If the points *B*, *D* do not coincide,

one of the angles *ABC*, *ADC* is an exterior angle and the other an interior angle of the triangle *BCD*;

therefore the angle *ABC* is not equal to the angle *ADC*, (I. Prop. 16.)

and therefore the arc *ABC* is not similar to the arc *ADC*. (Def. 4.)

It has now been proved that, if any straight line *ABD* meet the arcs in two points *B* and *D* which are not coincident,

the arcs cannot be similar.

Therefore, if the arcs be similar, every straight line drawn through *A* must meet the arcs in two coincident points, that is, the arcs *ABC*, *ADC* must coincide.

Wherefore, *two arcs of circles* &c.

EXERCISES.

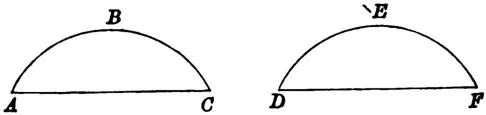
1. If on opposite sides of the same straight line there be two arcs of circles, which contain supplementary angles, the arcs are parts of the same circle.

2. Prove that, if two circles have three points in common, the circles are coincident.

PROPOSITION 24.

Similar arcs of circles, which have equal chords, are equal.

Let *ABC*, *DEF* be two similar arcs of circles, which have equal chords *AC*, *DF*:
it is required to prove that the arcs *ABC*, *DEF* are equal.



Proof. Because the chords *AC*, *DF* are equal, it is possible to shift the figure *ABC*, so that *AC* coincides with *DF*, *A* with *D*, and *C* with *F*, (I. Test of Equality, page 5)
and so that the arcs *ABC*, *DEF* are on the same side of *DF*.
If this be done,
the arc *ABC* must coincide with the arc *DEF*,
for the arcs *ABC*, *DEF* are similar,
and two arcs of circles, which have a common chord, and are on the same side of it, cannot be similar unless they coincide. (Prop. 23.)
Therefore the arcs *ABC*, *DEF* are equal.

Wherefore, *similar arcs of circles* &c.

EXERCISES.

1. If D be a point in the side BC of a triangle ABC whose sides AB, AC are equal, the circles described about the triangles ABD, ACD are equal.

2. Find a point P within an equilateral triangle ABC, such that the circles described about the triangles PBC, PCA, PAB may be all equal.

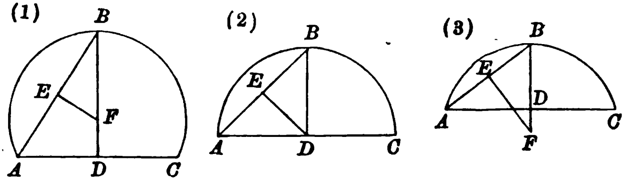
3. Find a point P in the plane of a triangle ABC such that the circles described about the triangles PBC, PCA, PAB may be equal.

PROPOSITION 25.

To find the centre of the circle, of which a given arc is a part.

Let ABC be a given arc:
it is required to find the centre of the circle, of which the arc ABC is a part.

CONSTRUCTION. Draw AC, and bisect it at D. (I. Prop. 10.)
At D draw DB at right angles to AC cutting the arc at B. (I. Prop. 11.)
Draw AB, bisect it at E, and at E draw EF at right angles to AB meeting BD or BD produced at F*;
then F is the centre required.



PROOF. Because DB bisects the chord AC at right angles,
DB passes through the centre; (Prop. 2.)
and because EF bisects the chord AB at right angles,
EF passes through the centre. (Prop. 2.)
Now two straight lines cannot intersect in more than one point. (I. Post. 1.)
Therefore F, the point of intersection of BD and EF, is the centre.

Wherefore, *the centre of the circle, of which the given arc ABC is a part, has been found.*

* The lines must meet, see Ex. 2, p. 51. In figure (2) F coincides with D.

PROPOSITION 25 A.

Equal circles have equal radii.

If two circles be equal, it is possible to shift one of them so as to coincide with the other. (I. Def. 21, page 13.)

Let this be done.

Then, because a circle cannot have more than one centre, (Prop. 1.)
the centres of the two coincident circles must be coincident:
and therefore all the radii of both circles are equal.

Wherefore, *equal circles* &c.

EXERCISES.

1. Having given two arcs of circles, shew how to find whether they are parts of the same circle.

2. Having given two arcs of circles, find whether they are parts of concentric circles.

3. Having given two arcs of circles, find whether one circle lies wholly within the other.

PROPOSITION 26.

In equal circles the arcs, on which equal angles at the centres stand, are equal; and the arcs, on which equal angles at the circumferences stand, are equal.

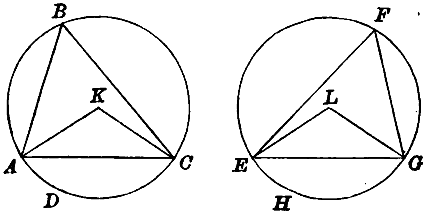
Let $ABCD$, $EFGH$ be two equal circles, and let AKC, ELG be two angles at the centres standing on the arcs ADC, EHG, and let ABC, EFG be two angles at the circumferences standing on the same arcs: and let

(1) the angles AKC, ELG be equal,

(2) the angles ABC, EFG be equal:

it is required in either case to prove that the arcs ADC, EHG are equal.

CONSTRUCTION. Draw AC, EG.



PROOF. Because the angle AKC is double of the angle ABC, (Prop. 20.)
and the angle ELG is double of the angle EFG, (Prop. 20.)
in case (1) because the angles AKC, ELG are equal,
the angles ABC, EFG are equal,
and in case (2) because the angles ABC, EFG are equal,
the angles AKC, ELG are equal.
Now because the circles are equal,
their radii AK, KC, EL, LG are equal. (Prop. 25 A.)
Therefore in both cases (1) and (2),
because in the triangles AKC, ELG,
AK is equal to EL,
KC to LG,
and the angle AKC to the angle ELG,
the triangles are equal in all respects; (I. Prop. 4.)
therefore AC is equal to EG.

And because the arcs ABC, EFG, which contain equal angles, have equal chords AC, EG,

the arcs ABC, EFG are equal: (Prop. 24.)

but the circles $ABCD$, $EFGH$ are equal;

therefore the remaining arcs ADC, EHG are equal.

Wherefore, *in equal circles* &c.

COROLLARY. *In the same circle the arcs, on which equal angles at the centres stand, are equal; and the arcs, on which equal angles at the circumferences stand, are equal.*

EXERCISES.

1. If PQ, RS be a pair of parallel chords in a circle, then the arcs PS, QR are equal, and the arcs PR, QS are equal.

2. A quadrilateral is inscribed in a circle, and two opposite angles are bisected by straight lines meeting the circumference in P and Q; prove that PQ is a diameter.

3. If through P any point on one of two circles, which intersect in A and B, the straight lines PA, PB be drawn and produced if necessary to cut the other circle in Q and R, the arc QR is of constant length.

4. The internal bisectors of the vertical angles of all triangles, on the same base and on the same side of it, which have equal vertical angles, pass through one fixed point and the external bisectors through another fixed point.

5. If through one of the points of intersection of two equal circles a straight line be drawn terminated by the circles, the straight lines joining its extremities with the other point of intersection are equal.

PROPOSITION 27.

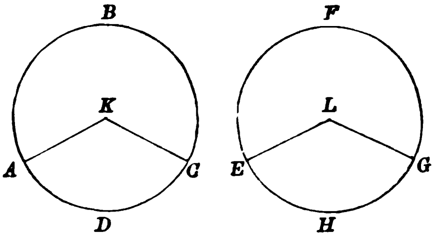
In equal circles, angles contained by arcs, which are of equal length, are equal.

Let *ABCD*, *EFGH* be equal circles, and let *ABC*, *EFG* be arcs of equal length:

it is required to prove that the arcs *ABC*, *EFG* contain equal angles.

Construction. Find the centres *K*, *L* of the circles *ABCD*, *EFGH*, (Prop. 5.)

and draw *AK*, *KC*, *EL*, *LG*.



Proof. Because the circles *ABCD*, *EFGH* are equal,

their radii *AK*, *KC*, *EL*, *LG* are equal. (Prop. 25 A.)

And because *AK* is equal to *EL*,

it is possible to shift the figure *ABCDK* so that *A* coincides with *E*, and *K* with *L*, and so that the parts of the arcs *ABC*, *EFG* near *E* are on the same side of *EL*.

If this be done,

because the radii of the circles are equal and their centres coincide,

the circles must coincide;

and because the circles coincide, and the parts of the arcs *ABC*, *EFG* near *E* are on the same side of *EL*, those parts of the arcs coincide;

and because the arcs are of equal length and have one extremity common,

therefore the other extremity must be common,

that is, the point *C* must coincide with the point *G*.

Therefore the arc *ABC* coincides with the arc *EFG*; and angles in the two arcs are then angles in the same arc and therefore equal. (Prop. 21.)

Wherefore, *in equal circles* &c.

Corollary. *In the same circle, angles contained by arcs, which are of equal length, are equal.*

EXERCISES.

1. The straight lines joining the extremities of two equal arcs of a circle are parallel or are equal.

Can they be both parallel and equal?

2. The straight lines bisecting any angle of a quadrilateral inscribed in a circle and the opposite exterior angle, meet on the circle.

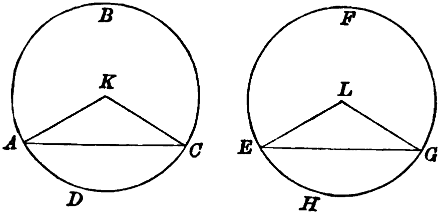
3. If from any point on a circle a chord and a tangent be drawn, the perpendiculars on them from the middle point of either of the arcs subtended by the chord are equal to one another.

PROPOSITION 28.

In equal circles, arcs cut off by chords, which are equal to one another, are of equal length, the greater equal to the greater and the less equal to the less.

Let *ABCD*, *EFGH* be equal circles, and let *AC*, *EG* be equal chords which cut off the two greater arcs *ABC*, *EFG*, and the two less arcs *ADC*, *EHG*:
it is required to prove that the arcs *ABC*, *EFG* are of equal length,
and that the arcs *ADC*, *EHG* are of equal length.

Construction. Find *K*, *L*, the centres of the circles *ABCD*, *EFGH*, (Prop. 5.)
and draw *AK*, *KC*, *EL*, *LG*.



Proof. Because the circles are equal,
their radii *AK*, *KC*, *EL*, *LG* are equal; (Prop. 25 A.)
and because in the triangles *AKC*, *ELG*,
KA is equal to *LE*,
KC to *LG*,
and *AC* to *EG*,
the triangles are equal in all respects; (I. Prop. 8.)
therefore the angle *AKC* is equal to the angle *ELG*.
But in equal circles the arcs, on which equal angles at the centres stand, are equal; (Prop. 26.)
therefore the arc *ADC* is equal to the arc *EHG*.
But the circle *ABCD* is equal to the circle *EFGH*;
therefore the arc *ABC* is equal to the arc *EFG*.

Wherefore, *in equal circles* &c.

Corollary. *In the same circle, arcs cut off by chords, which are equal to one another, are of equal length, the greater equal to the greater and the less equal to the less.*

EXERCISES.

1. A triangle is turned about its vertex till one of the sides passing through the vertex is in the same straight line as the other previously was. Prove that the line joining the vertex with the intersection of the two positions of the base, produced if necessary, bisects the angle between these two positions.

2. Find a point on one of two given equal circles, such that, if from it two tangents be drawn to the other circle, the chord joining the points of contact is equal to the chord of the first circle formed by joining its points of intersection with the two tangents produced.

Determine the conditions of the possibility of a solution of the problem.

PROPOSITION 29.

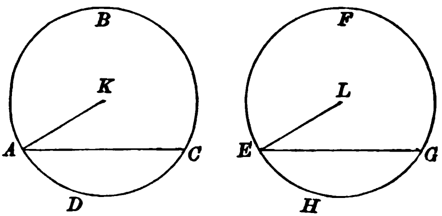
In equal circles, chords, by which arcs of equal length are subtended, are equal.

Let $ABCD$, $EFGH$ be equal circles, and let AC, EG be chords by which ADC, EHG, arcs of equal length are subtended:

it is required to prove that the chords AC, EG are equal.

Construction. Find K, L, the centres of the circles $ABCD$, $EFGH$, (Prop. 5.)

and draw AK, EL.



Proof. Because the circles $ABCD$, $EFGH$ are equal,
their radii AK, EL are equal. (Prop. 25 A.)
And because AK is equal to EL,
it is possible to shift the figure $ABCDK$ so that A coincides with E, and K with L, and so that the parts of the arcs ABC, EFG near E are on the same side of EL.
If this be done,
because the radii of the circles are equal and their centres coincide,
the circles must coincide;
and because the circles coincide, and the parts of the arcs ABC, EFG near E are on the same side of EL, those parts of the arcs coincide;
and because the arcs are of equal length and have one extremity common,
therefore the other extremity must be common,
that is, the point C must coincide with the point G.

Therefore the chord AC coincides with the chord EG and is equal to it.

Wherefore, *in equal circles* &c.

Corollary. *In the same circle, chords, by which arcs of equal length are subtended, are equal.*

EXERCISES.

1. If the diagonals of a quadrilateral inscribed in a circle bisect one another, the diagonals are diameters.

2. If two chords AP, AQ of a circle intersect at a constant angle at a fixed point A on the circle, the chord PQ always touches a concentric circle.

3. Two triangles are inscribed in a circle: if two sides of the one be parallel to two sides of the other, the third sides are equal.

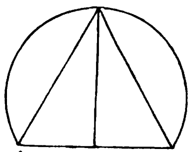
Is it necessary that they are parallel?

PROPOSITION 30.

To bisect a given arc of a circle.

Let ABC be the given arc:
it is required to bisect it.

CONSTRUCTION. Draw AC;
bisect it at D; (I. Prop. 10.)
at D draw DB at right angles to AC meeting the arc at B: (I. Prop. 11.)
the arc ABC is bisected as required at the point B.
Draw AB, BC.



PROOF. Because in the triangles ADB, CDB,
AD is equal to CD,
and DB to DB,
and the angle ADB to the angle CDB,
the triangles are equal in all respects; (I. Prop. 4.)
therefore AB is equal to CB.
But arcs cut off by equal chords are equal, the greater equal to the greater, and the less equal to the less; (Prop. 28, Coroll.)
and because BD, if produced, is a diameter, (Prop. 2.)
each of the arcs AB, CB is less than a semicircle,
and therefore the arc AB is the smaller of the two arcs cut off by the chord AB, and the arc CB the smaller of those cut off by the chord CB;
therefore the arc AB is equal to the arc CB.

Wherefore, *the given arc ABC is bisected at B.*

EXERCISES.

1. Find the triangle of maximum area which can be inscribed in a given circle having a given chord for one side.

2. Prove that the triangle of maximum area inscribed in a circle is equilateral.

3. The greatest quadrilateral which can be inscribed in a circle is a square.

4. Having given a regular polygon of any number of sides inscribed in a circle, inscribe a regular polygon of double the number of sides.

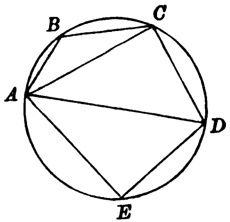
5. If ABC an arc of a circle less than a semicircle be bisected in B, and AB produced meet CD which is drawn at right angles to BC in D, and the tangents at A and C meet in E, then B, C, D, E lie on a circle.

PROPOSITION 31.

An angle in a semicircle is a right angle; an angle in an arc, which is greater than a semicircle, is less than a right angle; and an angle in an arc, which is less than a semicircle, is greater than a right angle.

Let *ABCDE* be a circle, of which *ABD* is a semicircle, *ADC* an arc greater than a semicircle, and *ABC* an arc less than a semicircle:
it is required to prove that the angle in the semicircle *ABD* is a right angle; that the angle in the arc *ADC* is less than a right angle, and that the angle in the arc *ABC* is greater than a right angle.

Construction. Take any point *B* in the arc *ABC* and any point *E* in the semicircle *AED* and draw *AB*, *BC*, *CD*, *DE*, *EA*, *AC*, *AD*.



Proof. Because *ACDE* is a quadrilateral inscribed in a circle, the sum of the angles *ACD*, *AED* is equal to two right angles. (Prop. 22.)
But because each of the angles *ACD*, *AED* is contained by a semicircle, the angle *ACD* is equal to the angle *AED*; (Prop. 27, Coroll.)
therefore each of them is a right angle.
Next, because the angle *ACD* of the triangle *ACD* is a right angle,
the angle *ADC* is less than a right angle, (I. Prop. 17.)
and it is an angle in the arc *ADC*.

Again, because $ABCD$ is a quadrilateral inscribed in a circle, the sum of the angles ABC, ADC is equal to two right angles. (Prop. 22.)

And the angle ADC has been proved to be less than a right angle;

therefore the angle ABC is greater than a right angle,

and it is an angle in the arc ABC.

Wherefore, *an angle in a semicircle* &c.

EXERCISES.

1. The circles described on two equal sides of a triangle as diameters intersect at the middle point of the third side.

2. The circles described on any two sides of a triangle as diameters intersect on the third side.

3. Construct the rectangle one of whose diagonals is a straight line given in magnitude and position and the other of whose diagonals passes through a given point.

4. An angle BAC of constant magnitude turns round its apex A which is fixed. Prove that the line joining the feet of the perpendiculars from a fixed point O on AB, AC always touches a fixed circle.

5. If a circle A pass through the centre of a circle B, the tangents to B at the points of intersections of A and B intersect on the circle A.

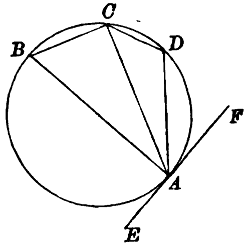
6. Two chords AB, CD of constant length placed in a circle subtend angles at the centre whose sum is equal to two right angles. If AC, BD intersect in P, the distances of P from the middle points of the chords will be independent of their relative positions.

PROPOSITION 32.

If a chord be drawn from the point of contact of a tangent to a circle, each of the angles which this chord makes with the tangent is equal to the angle in the alternate arc of the circle.*

Let $ABCD$ be a circle, and EAF be the tangent at a point A, and let AC be a chord drawn from the point A:
it is required to prove that the angle CAE is equal to the angle in the arc CDA, and the angle CAF to the angle in the arc CBA.

CONSTRUCTION. At A draw AB at right angles to EAF, cutting the circle again at B; (I. Prop. 11.)
take any point D in the arc ADC and draw BC, CD, DA.



PROOF. Because AB is drawn at right angles to EAF,
AB passes through the centre; (Prop. 19.)
therefore $BCDA$ is a semicircle; (Prop. 1 A.)
and the angle BCA is a right angle; (Prop. 31.)
therefore the sum of the other two angles BAC, CBA of the triangle ABC is equal to a right angle. (I. Prop. 32.)
And the sum of the angles BAC, CAF, that is, the angle BAF, is a right angle;
therefore the sum of the angles BAC, CAF is equal to the sum of the angles BAC, CBA.

* The *alternate* arc is the name generally given to the arc which lies on the side of the chord opposite to the angle spoken of.

Take away the common angle BAC;
then the angle CAF is equal to the angle CBA,
which is an angle in the arc ABC.

Again, because the sum of the angles ABC, ADC is equal to two right angles, (Prop. 22.)
and the sum of the angles CAF, CAE is equal to two right angles; (I. Prop. 13.)
therefore the sum of the angles CAF, CAE is equal to the sum of the angles ABC, ADC;
and it has been proved that the angle CAF is equal to the angle ABC;
therefore the angle CAE is equal to the angle ADC,
which is an angle in the arc ADC.

Wherefore, *if a chord be drawn* &c.

The theorem of Proposition 32 follows immediately from the theorem of Proposition 21, if we consider the tangent at a point as the limiting position of a chord drawn through the point. (See page 217.)

For the angle between the tangent AF in the diagram of Proposition 32 and the chord AC is the angle between two chords of the arc ABC, one an indefinitely short one drawn from a point indefinitely near A to A and the other drawn from the same point to C, and is therefore an angle in the arc ABC and therefore equal to the angle ABC. (Prop. 21.)

EXERCISES.

1. If two circles touch each other, any straight line drawn through the point of contact will cut off similar segments.

2. On the same side of portions AB, AC of a straight line ABC similar segments of circles are described: prove that the circles touch one another.

3. If two circles OPQ, Opq touch at O, and OPp, OQq be straight lines, the chords PQ, pq are parallel.

4. If a straight line cut two circles which touch at O, in the points P, Q, and p, q, the angles POp, QOq are either equal or supplementary.

5. ABC is a triangle inscribed in a circle, and from any point D in BC a straight line DE is drawn parallel to CA and meeting the tangent at A in E; shew that a circle may be described round $AEBD$.

PROPOSITION 33.

To describe on a given finite straight line an arc of a circle containing an angle equal to a given angle.

Let AB be the given straight line, and C the given angle:

it is required to describe on AB an arc of a circle containing an angle equal to the angle C.

CONSTRUCTION. Bisect AB at E. (I. Prop. 10.)

If the angle C be a right angle, with E as centre and EA or EB as radius, describe a circle:

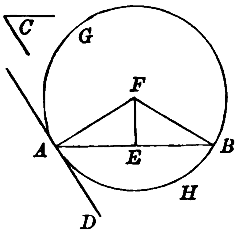
then the semicircle on either side of AB is an arc described as required. (Prop. 31.)

If the angle C be not a right angle, from A draw AD making the angle BAD equal to the angle C; (I. Prop. 23.) and at A, E draw AF, EF at right angles to AD, AB respectively, (I. Prop. 11.)

and let them meet at F*.

Draw FB, and with F as centre and FA as radius describe the circle AGH:

this circle passes through B and the arc AGB on the side of AB away from AD is an arc described as required.



PROOF. Because in the triangles FEA, FEB,

EA is equal to EB,

and FE to FE,

and the angle FEA to the angle FEB,

the triangles are equal in all respects; (I. Prop. 4.)

therefore FA is equal to FB.

* The lines must meet. See Ex. 2, p. 51.

Therefore the circle *AGH* passes through *B*.

Again, because *AD* is drawn from *A* at right angles to the radius *AF*,

AD touches the circle; (Prop. 16.)

and because the chord *AB* is drawn from the point of contact of the tangent *AD*,

the angle in the alternate arc *AGB* is equal to the angle *DAB*, (Prop. 32.)

that is, to the angle *C*.

Wherefore, *on the given straight line AB the arc AGB has been described containing an angle equal to the given angle C.*

EXERCISES.

1. Find a point at which each of two given finite straight lines subtends a given angle.

2. Construct a triangle, having given the base, the vertical angle, and the foot of the perpendicular from the vertex on the base.

3. Having given the base and the vertical angle of a triangle, construct the triangle which will have the maximum area.

4. Find a point *O* within a given triangle *ABC*, so that, if *AO*, *BO*, *CO* be joined, the angles *OAB*, *OBC*, *OCA* shall be all equal.

5. Construct a triangle, having given the base, the vertical angle, and the altitude.

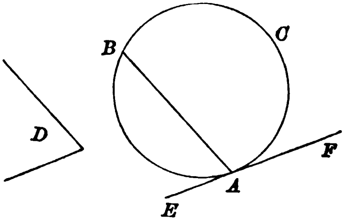
6. Find the locus of a point at which two given equal straight lines *AB*, *BC* subtend equal angles.

PROPOSITION 34.

To cut off from a given circle an arc containing an angle equal to a given angle.

Let ABC be the given circle and D the given angle: it is required to cut off from the circle ABC an arc containing an angle equal to the angle D.

Construction. Take any point A on the circle, and through A draw the straight line EAF to touch the circle at A. (Prop. 17.)
From A draw AB making the angle BAE equal to the angle D,
and cutting the circle again at B: (I. Prop. 23.)
then the arc ACB, on the side of AB away from E, is an arc cut off as required.



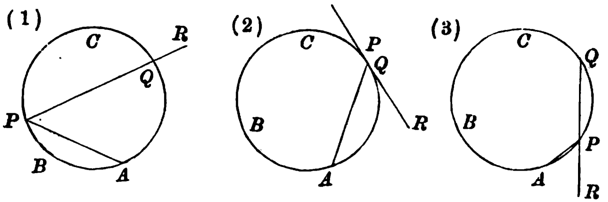
Proof. Because the chord AB is drawn from the point of contact A of the tangent EAF,
the angle EAB is equal to the angle in the alternate arc ACB. (Prop. 32.)
And the angle EAB is equal to the angle D;
therefore the angle in the arc ACB is equal to the angle D.

Wherefore, *the arc ACB has been cut off from the given circle ABC containing an angle equal to the given angle D.*

Through a given point *two* chords can be drawn which will cut off arcs containing an angle equal to a given angle.

Outline of Alternative Construction.

Through A draw any chord AP, and from P draw PR making the angle RPA equal to the given angle D, and cutting the circle again at Q.



It may be proved that the arc ABQ, measured from A on the side of AP opposite to that on which PR is drawn, is an arc cut off as required.

EXERCISES.

1. In a given circle inscribe an equiangular triangle.

2. Inscribe in a given circle a triangle, so that one angle may be a half of a second angle and a third of the third angle.

3. Inscribe in a given circle a right-angled triangle, so that one of its acute angles may be three times the other.

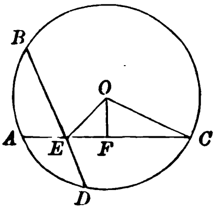
PROPOSITION 35.

If two chords of a circle intersect at a point within the circle, the rectangle contained by the segments of one chord is equal to the rectangle contained by the segments of the other chord.

Let $ABCD$ be a circle and AC, BD two chords intersecting at the point E within the circle:
it is required to prove that the rectangle contained by AE, EC is equal to the rectangle contained by BE, ED.

CONSTRUCTION. If the point E be the centre, it is clear that AE, EC, BE, ED are equal, each being a radius, and that the rectangles AE, EC and BE, ED, each of which is equal to the square on a radius, are equal.

If E be not the centre, find O the centre; (Prop. 5.)
draw OF at right angles to AC, (I. Prop. 12.)
and draw OE, OC.



PROOF. Because OF is drawn from the centre at right angles to the chord AC,
therefore AF is equal to FC. (Prop. 4.)
And because EC is the sum of FC, EF,
and AE is the difference of AF, EF, that is, of FC, EF,
therefore the rectangle AE, EC is equal to the difference of the squares on FC, EF. (II. Prop. 5.)

But because the angles at F are right angles,
the sum of the squares on OF, FC is equal to the square on OC,
and the sum of the squares on OF, EF is equal to the square on OE; (I. Prop. 47.)

therefore the difference of the squares on FC, EF is equal to the difference of the squares on OC, OE.

Therefore the rectangle AE, EC is equal to the difference of the squares on OC, OE.

Similarly it can be proved that the rectangle BE, ED is equal to the difference of the squares on OB, OE,

that is, is equal to the difference of the squares on OC, OE, since OC is equal to OB.

Therefore the rectangle AE, EC is equal to the rectangle BE, ED.

Wherefore, *if two chords of a circle* &c.

There are two special cases which should be noticed by the student, one case, when the points E, F coincide, i.e. when one chord bisects the other; the other case, when the points O, F coincide, i.e. when one chord is a diameter.

In Proposition 35 the distances between the ends of a chord and a point in the chord are spoken of as *segments of the chord*. In Proposition 36 it will be noticed that the expression *segments of a chord* has been used of the distances between the ends of the chord and a point taken in the chord produced. In the first case the chord is equal to the sum of the segments, in the second to the difference of the segments.

EXERCISES.

1. Prove the converse of Proposition 35, i.e. that, if AC, BD be two straight lines intersecting at E such that the rectangles AE, EC, and BE, ED are equal, then A, B, C, D lie on a circle.

2. If through any point in the common chord of two intersecting circles there be drawn any two other chords, one in each circle, their four extremities all lie on a circle.

3. Draw through a given point within a circle a chord, one of whose segments shall be four times as long as the other. When is this possible?

4. Divide a given straight line into two parts, so that the rectangle contained by the parts may be equal to a given rectangle.

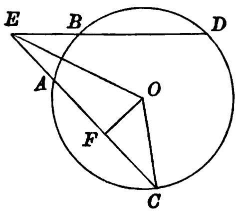
5. A, B, C are three points on a circle, D is the middle point of BC and AD produced meets the circle in E: prove that the sum of the squares on AB, AC is double of the rectangle AD, AE.

PROPOSITION 36.

If two chords of a circle when produced intersect at a point without the circle, the rectangle contained by the segments of one chord is equal to the rectangle contained by the segments of the other chord.

Let $ABDC$ be a circle and let CA, DB be two chords, which intersect, when produced beyond A and B, at the point E without the circle:
it is required to prove that the rectangle contained by EA, EC is equal to the rectangle contained by EB, ED.

CONSTRUCTION. Find O the centre; (Prop. 5.)
draw OF at right angles to AC, (I. Prop. 12.)
and draw OE, OC.



PROOF. Because OF is drawn from the centre at right angles to the chord AC,
therefore AF is equal to FC. (Prop. 4.)
And because EC is the sum of EF, FC,
and EA is the difference of EF, AF, that is, of EF, FC,
therefore the rectangle EA, EC is equal to the difference of the squares on EF, FC. (II. Prop. 6.)
But because the angles at F are right angles,
the sum of the squares on OF, FE is equal to the square on OE,
and the sum of the squares on OF, FC is equal to the square on OC; (I. Prop. 47.)
therefore the difference of the squares on EF, FC is equal to the difference of the squares on OE, OC.

Therefore the rectangle EA, EC is equal to the difference of the squares on OE, OC.

Similarly it can be proved that the rectangle EB, ED is equal to the difference of the squares on OE, OD, that is, is equal to the difference of the squares on OE, OC, since OD is equal to OC.

Therefore the rectangle EA, EC is equal to the rectangle EB, ED.

Wherefore, *if two chords of a circle* &c.

There are two special cases which should be noticed by the student, one case, when the points O, F coincide, i.e. when one chord is a diameter; the other case, when the points B, D coincide, i.e. when one chord is a tangent. The statement of the theorem in the latter case appears in the Corollary.

COROLLARY.

If a chord of a circle be produced to any point, the rectangle contained by the segments of the chord is equal to the square on the tangent drawn to the circle from the point.

This result is seen at once on considering the tangent as the limiting position of the secant.

EXERCISES.

1. Prove the converse of Proposition 36, i.e. that, if EAC, EBD be two straight lines intersecting at E such that the rectangles EA, EC and EB, ED are equal, then A, B, C, D lie on a circle.

2. If two circles intersect each other, their common chord bisects their common tangents.

3. From a given point as centre describe a circle cutting a given straight line in two points, so that the rectangle contained by their distances from a given point in the straight line may be equal to a given square.

4. If ABC be a triangle and D a point in AC such that the angle ABD is equal to the angle ACB, then the rectangle AC, AD is equal to the square on AB.

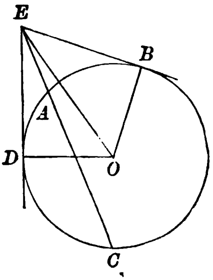
5. If from each of two given points, a pair of tangents be drawn to a given circle, the middle points of the chords joining the points of contact of each pair of tangents lie on the circumference of a circle passing through the two given points.

PROPOSITION 37.

If from an external point there be drawn to a circle two straight lines, one of which cuts the circle in two points and the other meets it, and if the rectangle contained by the segments of the chord on the line which cuts the circle be equal to the square on the line which meets the circle, the line which meets the circle is a tangent to it.

Let $ABCD$ be a circle and E an external point; and let EAC be a straight line cutting the circle at A, C and EB a straight line meeting it at B, such that the rectangle contained by EA, EC is equal to the square on EB: it is required to prove that EB touches the circle.

CONSTRUCTION. Find the centre O; (Prop. 5.)
from E draw ED to touch the circle at D; (Prop. 17.)
and draw OB, OD, OE.



PROOF. Because ED is a tangent, and OD is the radius,
the angle EDO is a right angle. (Prop. 18.)
Because EAC cuts the circle and ED touches it,
the rectangle EA, EC is equal to the square on ED;
(Prop. 36, Coroll.)
and the rectangle EA, EC is equal to the square on EB;
therefore the square on EB is equal to the square on ED;
therefore EB is equal to ED.

Again, because in the triangles EBO, EDO,
EB is equal to ED,
and OB to OD,
and OE to OE,
the triangles are equal in all respects; (I. Prop. 8.)
therefore the angle EBO is equal to the angle EDO.
But EDO is a right angle;
therefore the angle EBO is a right angle.
And because BE is at right angles to the radius OB,
BE touches the circle. (Prop. 16.)

Wherefore, *if from an external point* &c.

EXERCISES.

1. If three circles meet two and two, the common chords of each pair meet in a point.

2. If three circles touch two and two, the tangents at the points of contact meet at a point.

3. If the tangents drawn to two intersecting circles from a point be equal, the common chord of the circles passes through the point.

4. Describe a circle which shall touch a given straight line at a given point, and shall cut off from another given straight line a chord of a given length.

5. On OP, the straight line drawn from a given point O to P a point on a given straight line, a point Q is taken such that the rectangle OP, OQ is constant: prove that the locus of Q is a circle.

6. On OP, a chord of a given circle drawn from a given point O, a point Q is taken such that the rectangle OP, OQ is constant: prove that the locus of Q is a straight line.

PROPOSITION 37 A.

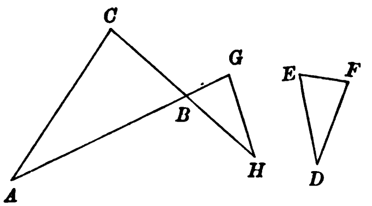
If two triangles be equiangular to one another, the rectangle contained by any side of the one and any side of the other is equal to the rectangle contained by the corresponding sides.*

Let *ABC*, *DEF* be two triangles which are equiangular to one another, having the angles at *A*, *B*, *C* equal to the angles at *D*, *E*, *F* respectively:

it is required to prove that the rectangle *AB*, *EF* is equal to the rectangle *BC*, *DE*.

CONSTRUCTION. In *AB*, *CB* produced beyond *B*, take points *G*, *H* such that *BG* is equal to *EF*, and *BH* to *ED*; (I. Prop. 3.)

and draw *GH*.



PROOF. Because the angle *GBH* is equal to the angle *CBA*, (I. Prop. 15.)

and the angle *FED* is equal to the angle *CBA*,

the angle *GBH* is equal to the angle *FED*.

Because in the triangles *BGH*, *EFD*,

BG is equal to *EF* and *BH* to *ED*,

and the angle *GBH* is equal to the angle *FED*,

the triangles are equal in all respects: (I. Prop. 4.)

therefore the angle *BGH* is equal to the angle *EFD*;

but the angle *EFD* is equal to the angle *BCA*,

therefore the angle *AGH* is equal to the angle *ACH*;

therefore the points *A*, *C*, *G*, *H* lie on a circle. (Prop. 21, Coroll.)

Therefore the rectangle *AB*, *BG* is equal to the rectangle *CB*, *BH*; (Prop. 35.)

that is, the rectangle *AB*, *EF* is equal to the rectangle *BC*, *DE*.

Wherefore, *if two triangles* &c.

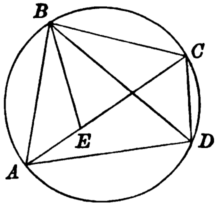
* In two triangles which are equiangular to one another, two sides are said to correspond when they are opposite to equal angles.

PROPOSITION 37 B.

The rectangle contained by the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the rectangles contained by pairs of opposite sides.*

Let $ABCD$ be a quadrilateral inscribed in a circle and AC, BD be its diagonals:
it is required to prove that the rectangle AC, BD is equal to the sum of the rectangles AB, CD and BC, AD.

CONSTRUCTION. From B in BA, on the same side of BA as CD, draw BE making the angle ABE equal to the angle CBD, and meeting AC in E. (I. Prop. 23.)



PROOF. Because the angle BAC is equal to the angle BDC, (Prop. 21.)
and the angle ABE is equal to the angle DBC, (Constr.)
therefore the triangles ABE, DBC are equiangular to one another; (I. Prop. 32.)
therefore the rectangle AB, CD is equal to the rectangle AE, BD. (Prop. 37 A.)
Again, because the angle ABE is equal to the angle DBC,
the angle ABD is equal to the angle EBC;
and the angle BDA is equal to the angle BCA (i.e. BCE), (Prop. 21.)
therefore the triangles ABD, EBC are equiangular to one another; (I. Prop. 32.)
therefore the rectangle AD, BC is equal to the rectangle EC, BD;
but it has been proved that
the rectangle AB, CD is equal to the rectangle AE, BD.
Therefore the sum of the rectangles AB, CD and AD, BC is equal to the sum of the rectangles AE, BD and EC, BD,
that is, to the rectangle AC, BD. (II. Prop. 1.)

Wherefore, *the rectangle contained* &c.

* This theorem is attributed to Ptolemy, a Greek geometer of Alexandria, who died about A.D. 160.

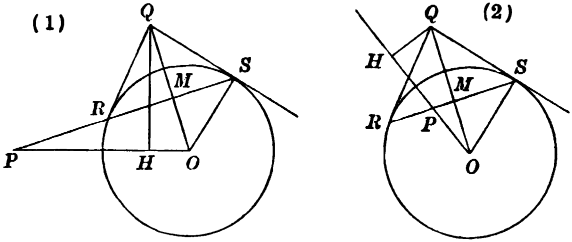
ADDITIONAL PROPOSITION.

If a straight line be drawn through a given point to cut a given circle, the intersection of the tangents at the two points of section always lies on a fixed straight line.*

Let PRS be any straight line drawn through a given point P to cut a given circle, whose centre is O, in R and S.

Let QR, QS be the tangents at R, S.

Draw OQ intersecting RS in M; and draw OP, and draw QH perpendicular to OP or OP produced.



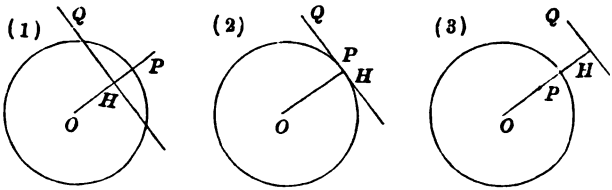
Because the angle at M is a right angle (Ex. 3, page 217),
and the angle at H is a right angle,
the points P, H, M, Q lie on a circle;
fig. 1 (Prop. 21, Coroll.) and fig. 2 (Prop. 22, Coroll.)
therefore the rectangle OH, OP is equal to the rectangle OM, OQ. (Prop. 36.)

But because QS is a tangent at S,
the angle OSQ is a right angle, (Prop. 18.)
and the angle at M is a right angle,
therefore the rectangle OM, OQ is equal to the square on OS; (I. Prop. 47.)

Therefore the rectangle OP, OH is equal to the square on OS.
But OP and OS are both constants,
therefore OH is a constant,
and the point Q always lies on a fixed straight line,
i.e. the line drawn through the fixed point H at right angles to OP.

* This line is called the polar of the given point, and the point is called the pole of the line with respect to the circle.

It has now been proved that, if a point and a straight line be such that the straight line joining the centre of a circle to the point is at right angles to the line, and the rectangle contained by the distances of the point and the line from the centre is equal to the square on the radius of the circle, the point is **the pole of the line**, and the line **the polar of the point** with respect to the circle.



In the diagram *O* is the centre of the circle: *H* is a point in *OP* such that the rectangle *OP*, *OH* is equal to the square on the radius, and *HQ* is at right angles to *OP*.

P is **the pole** of *HQ*, and *HQ* is **the polar** of *P*.

It will be observed that,

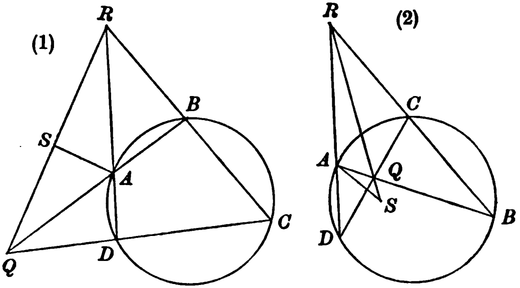
if *P* be without the circle (fig. 1), the polar cuts the circle:
if *P* be on the circle (fig. 2), the polar is the tangent to the circle,
and if *P* be within the circle (fig. 3), the polar does not cut the circle.

ADDITIONAL PROPOSITION.

If a quadrilateral be inscribed in a circle, the square on the straight line joining the points of intersection of opposite sides is less than the sum of the squares on the straight lines joining those points to the centre of the circle by twice the square on the radius of the circle.

Let $ABCD$ be a quadrilateral inscribed in a circle whose centre is O; and let the sides AB, CD meet in Q and the sides AD, BC in R.

Draw QR and draw AS making the angle RAS equal to the angle RQD and meeting RQ in S. (I. Prop. 23.)



Because in the triangles RAS, RQD,

the angle RAS is equal to the angle RQD,

the angle RSA is equal to the angle RDQ; (I. Prop. 32.)

therefore the points S, A, D, Q lie on a circle.

(Prop. 21 or 20, Coroll.)

Therefore the rectangle RS, RQ is equal to the rectangle RA, RD. (Prop. 36.)

Also it can be proved that the points R, S, A, B lie on a circle.

Therefore the rectangle QS, QR is equal to the rectangle QA, QB. (Prop. 35 or 36.)

Therefore the square on QR,

in figure (1), being the sum of the rectangles RS, RQ and QS, QR, is equal to the sum of the rectangles RA, RD and QA, QB;

and in figure (2), being the difference of the rectangles RS, RQ, and QS, QR, is equal to the difference of the rectangles RA, RD and QA, QB.

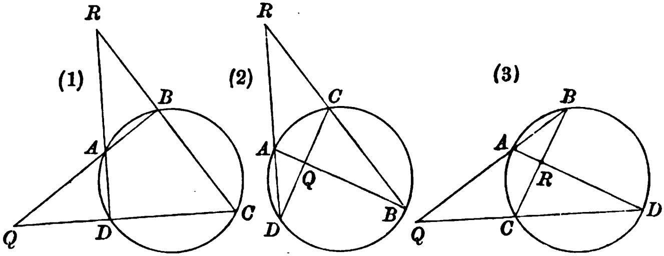
Since the rectangle RA, RD is equal to the difference of the squares on RO and the radius, (Prop. 36.)

and the rectangle QA, QB in figure (1) is equal to the difference of the squares on QO and the radius, (Prop. 36.)
and in figure (2) is equal to the difference of the squares on the radius and QO; (Prop. 35.)
it follows that in both cases
the square on QR is less than the sum of the squares on QO, RO by twice the square on the radius.

ADDITIONAL PROPOSITION.

If one pair of opposite sides of a quadrilateral inscribed in a circle intersect at a fixed point, the other pair of opposite sides intersect on a fixed straight line.*

Let $ABCD$ be a quadrilateral inscribed in a circle, whose centre is O; and let the sides AB, CD meet in Q and AD, BC in R.



Because Q and R are the intersections of opposite sides of a quadrilateral inscribed in the circle,
the square on QR is less than the squares on OQ, OR by twice the square on the radius; (Add. Prop. page 260.)
therefore the difference of the squares on QR, OQ is equal to the difference of the square on OR and twice the square on the radius, which is a constant, if the point R be fixed.

Therefore the locus of the point Q is a straight line.
(Ex. 2, page 125.)

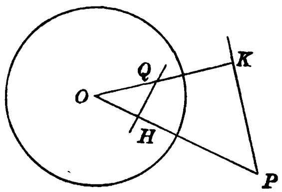
* We leave to the student as an exercise the proof that this line is the polar of the fixed point.

ADDITIONAL PROPOSITION.

If one point lie on the polar of another point, the second point lies on the polar of the first point.

Let P, Q be two points such that Q lies on the polar of P,
i.e. if QH be drawn perpendicular to OP,
the rectangle OH, OP is equal to the square on the radius.

CONSTRUCTION. Draw PK perpendicular to OQ. (I. Prop. 12.)



PROOF. Because the angles at H and K are right angles,
Q, K, P, H lie on a circle; (Prop. 22, Coroll.)
therefore the rectangle OQ, OK is equal to the rectangle OH, OP, (Prop. 36.)
and therefore to the square on the radius;
and KP is at right angles to OK;
therefore KP is the polar of Q,
or, in other words, P lies on the polar of Q.

EXERCISES.

1. Prove that the polar of a point without a circle is the straight line joining the points of contact of tangents drawn from the point to the circle.

2. If O be the centre of a circle, and the polar of a point P cut PO in H, and any straight line through P cut the circle in R and S, then the polar bisects the angle RHS.

3. If a straight line PQR cut a circle in Q and R and cut the polar of P in K, and if M be the middle point of QR, then the rectangles PQ, PR and PK, PM are equal.

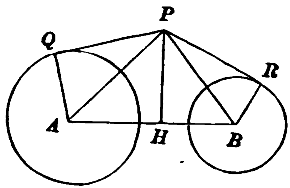
4. If P, Q, R, S be the points of contact of the sides AB, BC, CD, DA of a quadrilateral $ABCD$ with an inscribed circle, the straight lines AC, BD, PR, QS are concurrent.

5. Shew how to draw two tangents to a given circle from a given external point by means of straight lines only.

6. Shew how to draw a tangent to a given circle at a given point on it by means of straight lines only.

ADDITIONAL PROPOSITION.

The locus of a point from which tangents drawn to two given circles are equal is a straight line.*



Let P be a point such that PQ, PR tangents drawn to two given circles are equal.

Find the centres A, B of the circles; (Prop. 5.)
draw AB, AP, AQ, BP, BR, and draw PH perpendicular to AB. (I. Prop. 12.)

Because PQ, PR are tangents
the angles at Q and R are right angles.

Therefore the sum of the squares on PQ, AQ is equal to the square on AP, (I. Prop. 47.)
and the sum of the squares on PR, BR is equal to the square on BP;
therefore the difference of the squares on AQ, BR is equal to the difference of the squares on AP, BP.

But because the angles at H are right angles,
the difference of the squares on AP, BP
is equal to the difference of the squares on AH, HB.

Therefore the difference of the squares on AH, HB is equal to the difference of the squares on AQ, BR, which is a constant;
therefore H is a fixed point,
and the straight line HP on which P lies is drawn through H at right angles to AB the line of the centres, and is therefore a fixed straight line.

* This line is called the **Radical Axis** of the two circles. This name was given to the line by L. Gaultier de Tours, a French geometer. See *Journal de l'école Polytechnique*, tom. IX. p. 139 (1813).

EXERCISES.

1. Prove that the radical axis of two intersecting circles passes through their points of intersection.

2. What is the radical axis of two circles which touch each other?

3. Prove that the middle points of the four common tangents of two circles external to each other lie on a straight line.

4. Prove that the radical axes of three circles taken two and two together meet in a point*.

5. Shew how to draw the radical axis of two circles which do not meet.

6. Draw a circle passing through a given point and cutting two given circles so that its chords of intersection with the two circles may each pass through given points.

7. O is a fixed point outside a given circle: find a straight line such that each of the tangents drawn from any point P in that line to the circle shall be equal to PO.

8. Draw a straight line in a given direction so that chords cut from it by two given circles may be equal.

9. Prove that the difference of the squares of the tangents from any point to two circles is equal to twice the rectangle under the distance between their centres and the distance of the point from their radical axis.

10. Through two given points draw a circle to cut a given circle in such a way that the angle contained in the segment cut off the given circle may be equal to a given angle.

* This point is called **the Radical Centre** of the three circles.

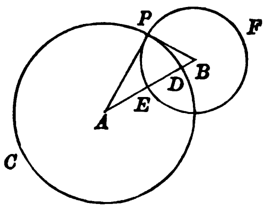
DEFINITION. *Two circles or other curves, which meet at a point, are said to meet at the angle at which their tangents at the point meet.*

Two circles or other curves are said to be **orthogonal** *or to cut orthogonally at a point, when they intersect at right angles at the point.*

ADDITIONAL PROPOSITION.

If the square on the distance between the centres of two circles be equal to the sum of the squares on the radii, the circles are orthogonal.

Let A, B be the centres of two circles CPD, EPF, which intersect at P, and are such that the square on AB is equal to the sum of the squares on AP, BP.



Because the square on AB is equal to the sum of the squares on AP, BP,

the angle APB is a right angle. (I. Prop. 48.)

And BP is a radius of the circle EPF;

therefore AP touches the circle EPF.

Similarly it can be proved that BP touches the circle CPD;

therefore the circles CPD, EPF are orthogonal.

COROLLARY. *The radius of each of two orthogonal circles drawn to a point of intersection is a tangent to the other circle.*

EXERCISES.

1. A circle, which passes through a given point and cuts a given circle orthogonally, passes through a second fixed point.

2. Describe a circle to cut a given circle orthogonally at two given points.

3. Describe a circle through two given points to cut a given circle orthogonally.

4. Two chords AD, BC of a circle $ACDB$, of which AB is a diameter, intersect at E: a circle described round CDE will cut the circle $ACDB$ at right angles.

5. Two circles cut each other at right angles in A, B; P is any point on one of the circles, and the lines PA, PB cut the other circle in Q, R: shew that QR is a diameter.

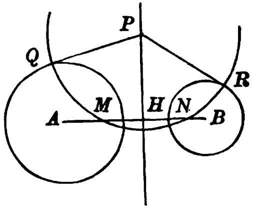
6. The internal and external bisectors of the vertical angle A of the triangle ABC meet the base in D and E respectively. Prove that the circles described about the triangles ABD and ABE cut at right angles, as also do those described about the triangles ACD and ACE.

ADDITIONAL PROPOSITION.

Every circle, which cuts two given circles orthogonally, has its centre on the radical axis of the given circles, and if it cut the straight line joining their centres, it cuts it in two fixed points.

Let A, B be the centres of two given circles and let P be the centre of a circle, which cuts the given circles orthogonally at Q and R.

Draw AB, PQ, PR, and draw PH perpendicular to AB.



Because the circles cut orthogonally at Q,

PQ is a tangent at Q.

(Add. Prop. page 266, Coroll.)

Similarly it can be proved that

PR is a tangent at R.

But PQ is equal to PR;

therefore P is a point on the radical axis of the given circles,

and therefore H is a fixed point. (Add. Prop. page 264.)

Next, let the circle whose centre is P cut the line AB in M, N.

Because the circles cut orthogonally at Q,

AQ is a tangent to the circle $QMNR$ at Q,

and therefore the square on AQ is equal to the rectangle AM, AN;

but because PH is at right angles to MN,

MH is equal to NH; (Prop. 4.)

and the rectangle AM, AN is equal to the difference of the squares on AH, MH; (II. Prop. 6.)

therefore the square on AQ is equal to the difference of the squares on AH, MH.

Now the lines AQ, AH are of constant length;

therefore MH (or NH) is of constant length.

Therefore the points M and N are at a constant distance from H, which is a fixed point;

therefore the points M and N are fixed points.

We leave it to the student as an exercise to prove that:

if the given circles be external to each other, the points M and N are real, one within each of the given circles;

if the circles touch externally, M and N coincide with the point of contact;

if the circles intersect, the circle, whose centre is P, does not intersect the line AB in real points;

if one circle touch the other internally, the points are again real, and they coincide with the point of contact;

if one circle lie wholly within the other, the points M and N are both real, one within both circles and the other without both circles.

EXERCISES.

1. Draw a circle to cut three given circles orthogonally.

2. Prove that every pair of circles, which cut two given circles orthogonally, has the same radical axis.

3. Of four given circles three have their centres in the same straight line, and the fourth cuts the other three orthogonally; prove that the radical axis of each pair of the three circles is the same.

4. $ABCD$ is a quadrilateral inscribed in a circle; the opposite sides AB and DC are produced to meet at F; and the opposite sides BC and AD at E: shew that the circle described on EF as diameter cuts the circle $ABCD$ at right angles.

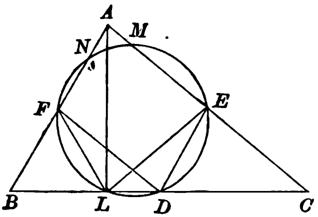
5. Find a point such that its polar with respect to each of two given circles is the same.

ADDITIONAL PROPOSITION.

The middle points of the sides of a triangle and the feet of the perpendiculars from the angular points on the opposite sides lie on a circle.

Let *D*, *E*, *F* be the middle points of the sides *BC*, *CA*, *AB* of a triangle *ABC*, and *L*, *M*, *N* the feet of the perpendiculars on them from *A*, *B*, *C*.

Draw *FL*, *LE*, *FD*, *DE*.



Then because *ALB* is a right-angled triangle, and *F* is the middle point of *AB*,

FL is equal to *FA*; (Ex. 7, page 87.)

therefore the angle *FLA* is equal to the angle *FAL*. (I. Prop. 5.)

Similarly it can be proved that the angle *ALE* is equal to the angle *LAE*;

therefore the angle *FLE* is equal to the angle *BAC*.

Again because *FD* is parallel to *AC*

and *DE* to *BA*, (Add. Prop. page 101.)

FAED is a parallelogram,

and the angle *FDE* is equal to the angle *BAC*; (I. Prop. 34.)

therefore the angle *FLE* is equal to the angle *FDE*.

Therefore *L*, *D*, *E*, *F* lie on a circle. (Prop. 21, Coroll.)

Similarly it can be proved that *M*, *D*, *E*, *F* lie on a circle,

and that *N*, *D*, *E*, *F* lie on a circle.

But only one circle can be described through the three points *D*, *E*, *F*;

therefore these three circles are coincident.

Therefore the six points *L*, *M*, *N*, *D*, *E*, *F* lie on a circle.

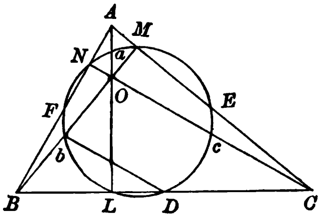
ADDITIONAL PROPOSITION.

The circle through the middle points of the sides of a triangle passes through the middle points of the straight lines joining the angular points of the triangle to the orthocentre.

Let D, E, F be the middle points of the sides BC, CA, AB of a triangle ABC.

Draw AL, BM, CN perpendicular to BC, CA, AB, intersecting at O. (Add. Prop. page 95.)

Bisect AO, BO, CO at a, b, c.



In the triangle OBC, D, c, b are the middle points of the sides, and L, M, N are the feet of the perpendiculars from the vertices on the opposite sides;

therefore L, D, c, M, N, b lie on a circle. (Add. Prop. page 270.)

Similarly it can be proved that

L, c, E, M, a, N lie on a circle,

and that L, M, a, N, F, b lie on a circle.

But only one circle can be described through the three points L, M, N;

therefore these circles are coincident.

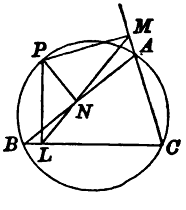
Therefore the nine points L, D, c, E, M, a, N, F, b lie on a circle *.

* This circle is called the **Nine Point Circle** of the triangle.

ADDITIONAL PROPOSITION.

The feet of the perpendiculars drawn from any point on a circle to the three sides of a triangle inscribed in the circle lie on a straight line.*

Let *ABC* be a triangle and *PL*, *PM*, *PN* be the perpendiculars from a point *P* on the circle *ABC* to *BC*, *CA*, *AB*.



Draw *LN*, *NM*.

Because *PLB*, *PNB* are right angles,

a circle can be described about *PNLB*; (III. Prop. 21, Coroll.)

therefore the angles *PNL*, *PBL* are supplementary. (III. Prop. 22.)

But the angles *PAC*, *PBC* are supplementary; (III. Prop. 22.)

therefore the angle *PNL* is equal to the angle *PAC*.

Again because *PMA*, *PNA* are right angles,

a circle can be described about *PNAM*; (III. Prop. 22, Coroll.)

therefore the angle *PNM* is equal to the angle *PAM*.

Therefore the sum of the angles *PNL*, *PNM* is equal to the sum of the angles *PAC*, *PAM*,

that is, to two right angles. (I. Prop. 13.)

Therefore *LN*, *NM* are in the same straight line. (I. Prop. 14.)

EXERCISES.

1. If *PL*, *PM*, *PN* be the perpendiculars drawn from *P* a point on the circle *ABC* to the sides *BC*, *CA*, *AB* of an inscribed triangle, and straight lines *Pl*, *Pm*, *Pn* be drawn to meet the sides in *l*, *m*, *n* such that the angles *LPl*, *MPm*, *NPn* are equal and measured in the same sense, then *l*, *m*, *n* are collinear.

2. *P* is a point on the circle circumscribing the triangle *ABC*. The pedal line of *P* cuts *AC* and *BC* in *M* and *L*. *Y* is the foot of the perpendicular from *P* on the pedal line. Prove that the rectangles *PY*, *PC*, and *PL*, *PM* are equal.

* This line is called **the Pedal Line**.

Its discovery is attributed to Dr Robert Simson, and it is in consequence also called **Simson's Line**.

BOOK IV.

DEFINITIONS.

DEFINITION 1.

A figure of five sides is called a **pentagon,**
one of six sides is called a **hexagon,**
one of eight sides is called an **octagon,**
one of ten sides is called a **decagon,**
one of twelve sides is called a **dodecagon***.

DEFINITION 2. *When each of the angular points of one rectilineal figure lies on one of the sides of a second rectilineal figure, and each of the sides of the second figure passes through one of the angular points of the first figure, the first figure is said to be* **inscribed in** *the second figure, and the second figure is said to be* **described about** *the first figure.*

* Derived from *πέντε* "five," *ἕξ* "six," *ὀκτώ* "eight," *δέκα* "ten," *δώδεκα* "twelve," respectively, and *γωνία* "an angle."

PROPOSITION 1.

To draw a chord of a given circle equal to a given straight line.

Let ABC be the given circle, and D the given straight line:

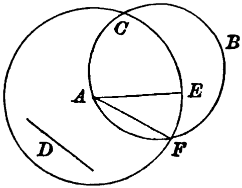
it is required to draw a chord of the circle ABC equal to D.

Construction. Take any point A on the circle ABC, and from A draw AE equal to D. (I. Prop. 2.)
If E lie on the circle ABC, what is required is done, for in the circle ABC the chord AE is drawn equal to D.

But if E do not lie on the circle ABC,
with A as centre and AE as radius describe the circle ECF cutting the circle ABC at F.

Draw AF.

AF is a chord drawn as required.



Proof. Because A is the centre of the circle ECF,

AF is equal to AE.

But AE is equal to D. (Constr.)

Therefore AF is equal to D,

and it is a chord of the circle ABC.

Wherefore, *a chord AF of the given circle ABC has been drawn equal to the given straight line D.*

It is clear that it is not possible to draw a chord of a given circle to be equal to a given straight line, if the given line be greater than the diameter of the circle (III. Prop. 8, Part 1); and further that, if a solution be possible, in general *two* chords can be drawn from a given point equal to the given line.

In the diagram, if the two circles intersect in C, the chord AC also is equal to the given line.

EXERCISES.

1. In a given circle draw a chord parallel to one given straight line and equal to another.

2. On a given circle find a point such that, if chords be drawn to it from the extremities of a given chord, their sum shall be equal to a given straight line.

How many solutions are there in the different cases which may occur?

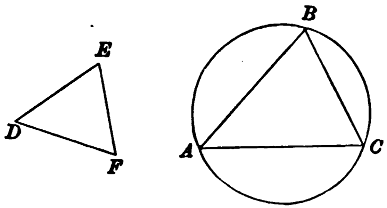
PROPOSITION 2.

To inscribe in a given circle a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle:
it is required to inscribe in the circle ABC a triangle equiangular to the triangle DEF.

CONSTRUCTION. Take any point A on the circle,
and through A draw the chord AB to cut off the arc ACB containing an angle equal to the angle DFE,
and through A draw the chord AC to cut off the arc ABC containing an angle equal to the angle DEF*.
(III. Prop. 34.)

Draw BC:
the triangle ABC is inscribed as required.



PROOF. Because the arc ACB contains an angle equal to the angle DFE, (Constr.)
and the angle ACB is contained by the arc ACB,
the angle ACB is equal to the angle DFE.
Similarly it can be proved that
the angle ABC is equal to the angle DEF.
And because the sum of three angles of a triangle is equal to two right angles, (I. Prop. 32.)
and the angles ACB, ABC are equal to the angles DFE, DEF respectively,

* It must be noticed that the arcs ACB, ABC are measured in opposite directions along the circumference from the point A.

the remaining angle BAC of the triangle ABC is equal to the remaining angle EDF of the triangle DEF;
therefore the triangle ABC is equiangular to the triangle DEF.

Wherefore, *a triangle ABC equiangular to the triangle DEF has been inscribed in the given circle ABC.*

Since the arc ABC may be measured in either direction along the circumference from A, we see that two triangles equiangular to a given triangle can be inscribed in a given circle so as to have a vertical angle equal to a given angle of the triangle at a given point on the circle, and that six triangles equiangular to a given triangle can be inscribed in a given circle, so as to have one of its vertical angles at the given point on the circle.

EXERCISES.

1. Prove that all triangles inscribed in the same circle equiangular to each other are equal in all respects.

2. The altitude of an equilateral triangle is equal to a side of an equilateral triangle inscribed in a circle described on one of the sides of the original triangle as diameter.

3. ABC, $A'B'C'$ are two triangles equiangular to each other inscribed in a circle $AA'BB'CC'$. The pairs of sides BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ intersect in a, b, c respectively.

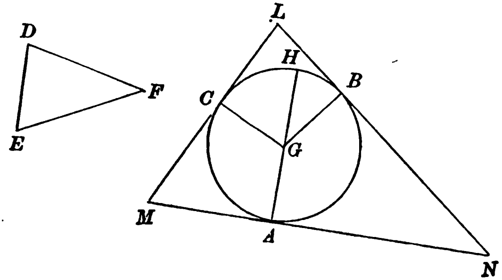
Prove that the triangle abc is equiangular to the triangle ABC.

PROPOSITION 3.

To describe about a given circle a triangle equiangular to a given triangle.

Let ABC be the given circle and DEF the given triangle:
it is required to describe about the circle ABC a triangle equiangular to the triangle DEF.

CONSTRUCTION. Find the centre G of the circle ABC, (III. Prop. 5.)
and draw any diameter HGA meeting the circle in A.
At G in GH draw the straight lines GB, GC on opposite sides of GH making the angles BGH, CGH equal to the angles EFD, DEF, (I. Prop. 23.)
meeting the circle in B, C.
Through A, B, C draw MAN, NBL, LCM at right angles to GA, GB, GC respectively: (I. Prop. 11.)
the triangle LMN is a triangle described as required.



PROOF. Because the sum of the angles of the quadrilateral $GBNA$ is equal to four right angles, (I. Prop. 32, Coroll.)
and two of the angles GAN, GBN are right angles,
the sum of the angles AGB, ANB is equal to two right angles.
But the sum of the angles AGB, HGB is equal to two right angles; (I. Prop. 13.)
therefore the sum of the angles AGB, ANB is equal to the sum of the angles AGB, HGB;
therefore the angle ANB is equal to the angle HGB,
that is, to the angle EFD.

Similarly it can be proved that

the angle *LMN* is equal to the angle *DEF*;

therefore the remaining angle *NLM* of the triangle *LMN* is equal to the remaining angle *EDF* of the triangle *DEF*. (I. Prop. 32.)

Therefore the triangle *LMN* is equiangular to the triangle *DEF*.

Again, because *MN* is drawn through *A* a point on the circle *ABC* at right angles to the radius *AG*,

MN touches the circle. (III. Prop. 16.)

Similarly it can be proved that *NL*, *LM* touch the circle.

Therefore the triangle *LMN* is described about the circle *ABC*.

Wherefore, *a triangle LMN equiangular to the given triangle DEF has been described about the given circle ABC.*

EXERCISES.

1. Prove that all triangles described about the same circle equiangular to each other are equal in all respects.

2. Describe a triangle about a given circle to have its sides parallel to the sides of a given triangle.

How many solutions are there?

3. The angles of the triangle formed by joining the points of contact of the inscribed circle of a triangle with the sides are equal to the halves of the supplements of the corresponding angles of the original triangle.

4. If *ABC*, *A′B′C′* be two equal triangles described about a circle in the same sense and the pairs of sides *BC*, *B′C′*; *CA*, *C′A′*; *AB*, *A′B′* meet in *a*, *b*, *c* respectively, *a*, *b*, *c* are equidistant from the centre of the circle.

PROPOSITION 4.

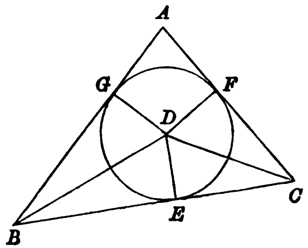
To inscribe a circle in a given triangle.

Let ABC be the given triangle:
it is required to inscribe a circle in the triangle ABC.

CONSTRUCTION. Bisect two of the angles ABC, BCA of the triangle ABC by BD, CD meeting at D, (I. Prop. 9.) and from D draw DE, DF, DG perpendicular to BC, CA, AB respectively. (I. Prop. 12.)

With D as centre and DE, DF, or DG as radius describe a circle:

it will be a circle described as required.



PROOF. Because in the triangles DEB, DGB,
the angle DBE is equal to the angle DBG, (Constr.)
and the angle DEB is equal to the angle DGB, (I. Prop. 10 B.)
and the side BD is common,
the triangles are equal in all respects; (I. Prop. 26, Part 2.)
therefore DE is equal to DG.

Similarly it can be proved that DE is equal to DF.

Therefore the three straight lines DE, DF, DG are equal to one another, and the circle described with D as centre, and DE, DF, or DG as radius passes through the extremities of the other two;

and touches the straight lines BC, CA, AB, because the angles at the points E, F, G are right angles, and the straight line drawn through a point on a circle at right angles to the radius touches the circle. (III. Prop. 16.)

Therefore the straight lines *AB*, *BC*, *CA* do each of them touch the circle, and therefore the circle is inscribed in the triangle *ABC*.

Wherefore, *the circle EFG has been inscribed in the given triangle ABC.*

EXERCISES.

1. The base of a triangle is fixed, and the vertex describes a circle passing through the extremities of the base: find the locus of the centre of the inscribed circle.

2. If a polygon be described about a circle, the bisectors of all its angles meet in a common point.

3. Describe a circle to touch a given circle and two given tangents to the circle.

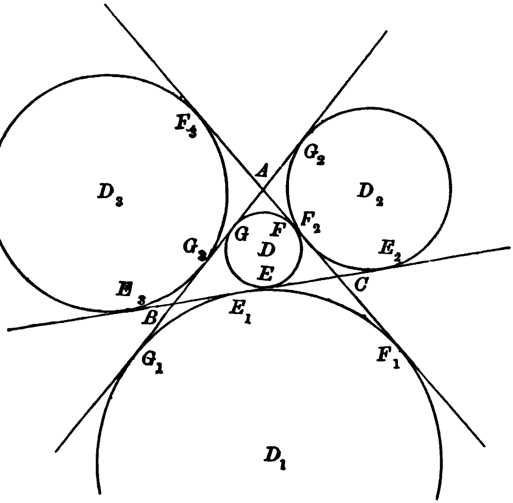
4. Construct a triangle, having given the base, the vertical angle and the radius of the inscribed circle.

5. Find the centre of a circle cutting off three equal chords from the sides of a triangle.

6. The triangle whose vertices are the three points of contact of the inscribed circle with the sides of a triangle, is always acute-angled.

It can be proved in the same manner as in Proposition 4 that, if the angles at B and C of the triangle be bisected *externally* by BD_1, CD_1, meeting at D_1, and perpendiculars, D_1E_1, D_1F_1, D_1G_1 be drawn, the circle described with D_1 as centre and either of the three lines D_1E_1, D_1F_1, D_1G_1 as radius will touch the three sides of the triangle. Such a circle satisfies the definition (III. Def. 10) of an inscribed circle.

The circles are however generally distinguished thus, the circle EFG, which lies wholly within the triangle ABC, is called **the inscribed circle**, whereas the circle $E_1F_1G_1$ is called **an escribed circle**, and is said to be **escribed beyond the side BC**, to distinguish it from the two other circles which can, in a similar manner, be escribed beyond CA and beyond AB respectively.



EXERCISES.

1. Prove that the radius of the inscribed circle of a triangle is less than the radius of any one of the escribed circles.

2. Prove that the greatest of the escribed circles of a triangle is that which is escribed beyond the greatest side, and the least, beyond the least side.

3. If the centres of the escribed circles of a triangle be joined, and the points of contact of the inscribed circle be joined, the two triangles so formed are equiangular to each other.

4. A circle touches the side BC of a triangle ABC and the other two sides produced: shew that the distance between the points of contact of the side BC with this circle and with the inscribed circle, is equal to the difference between the sides AB and AC.

5. Construct a triangle, having given its base, one of the angles at the base, and the distance between the centre of the inscribed circle and the centre of the circle touching the base and the sides produced.

6. Prove that, if A, B be two fixed points on a circle and P a variable point, the locus of the centre of each of the escribed circles of the triangle APB is a circle.

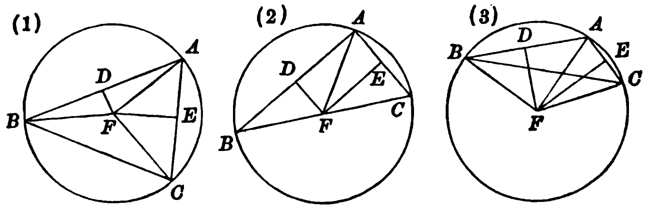
7. The centre of the inscribed circle of a triangle is the orthocentre of the triangle formed by the centres of the escribed circles.

PROPOSITION 5.

To describe a circle about a given triangle.

Let ABC be the given triangle:
it is required to describe a circle about the triangle ABC.

CONSTRUCTION. Bisect two of the sides AB, AC of the triangle ABC, at D, E, (I. Prop. 10.)
and draw DF, EF at right angles to AB, AC meeting at F. (I. Prop. 12.)
Draw FA, and with F as centre and FA as radius describe a circle:
this is a circle described as required.
F must lie either in BC (fig. 2) or not in BC (figs. 1 and 3).
If F do not lie in BC, draw FB, FC.



PROOF. Because in the triangles FDA, FDB,
AD is equal to BD, (Constr.)
and DF is equal to DF,
and the angle ADF is equal to the angle BDF, (Constr.)
the triangles are equal in all respects; (I. Prop. 4.)
therefore FA is equal to FB.

Similarly it can be proved that FA is equal to FC.
Therefore the circle described with F as centre and FA as radius passes through the points B and C, and is described about the triangle ABC.

Wherefore, *a circle ABC has been described about the given triangle ABC.*

The construction of Proposition 5 shews that only one circle can be described about a given triangle, a theorem which has already been established otherwise.

(III. Prop. 9, Coroll. 2.)

The circle *ABC* is often spoken of as the **circumscribed circle** of the triangle *ABC*.

Propositions 4 and 5 solve problems of the same nature; each shews how to describe a circle to satisfy three given conditions. The problem of Proposition 4 *to describe a circle to touch three given straight lines*, admits of 4 solutions; Proposition 5, *to describe a circle to pass through three given points*, admits of but a single solution.

A circle can be described to satisfy three (and not more than three) independent conditions, but it will be found that the solution is not always unique: if the problem be one which can be solved by geometrical methods, the number of solutions will be found to be 1 or 2 or $4=2\times 2$ or $8=2\times 2\times 2$ or some higher power of 2.

The number 2 occurs in one of its powers from the fact that at each step of the solution where choice is possible, the choice lies between the *two* intersections of a circle and a straight line or the *two* intersections of two circles.

If it be required to describe a circle to touch four or more given straight lines, or to pass through four or more given points, relations of some kind must exist between the positions of the lines or of the points in order that a solution may be possible.

EXERCISES.

1. Inscribe in an equilateral triangle another equilateral triangle having each side equal to a given straight line.

2. Shew how to cut off the corners of an equilateral triangle, so as to leave a regular hexagon.

3. The sides *AB*, *AC* of a triangle are produced and the exterior angles are bisected by straight lines meeting in *O*: if a circle be described about the triangle *BOC*, its centre will be on the circle described about the triangle *ABC*.

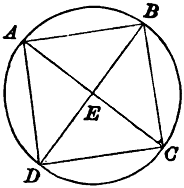
PROPOSITION 6.

To inscribe a square in a given circle.

Let $ABCD$ be the given circle:
it is required to inscribe a square in the circle $ABCD$.

CONSTRUCTION. Find the centre E of the circle $ABCD$, (III. Prop. 5.)
and draw two diameters AEC, BED at right angles to one another. (I. Prop. 11.)

Draw AB, BC, CD, DA:
the quadrilateral $ABCD$ is a square inscribed as required.



PROOF. Because the angle BEC is double of the angle BAC,
and the angle AED is double of the angle ACD, (III. Prop. 20.)
and the angle BEC is equal to the angle AED, (I. Prop. 15.)
therefore the angle BAC is equal to the angle ACD.
And because AC meeting AB, CD makes the alternate angles BAC, ACD equal,
AB, CD are parallel. (I. Prop. 27.)
Similarly it can be proved that AD, BC are parallel.
Therefore the quadrilateral $ABCD$ is a parallelogram.

Again, because ABC is an angle in a semicircle ABC,
the angle ABC is a right angle. (III. Prop. 31.)
Therefore the parallelogram $ABCD$ is a rectangle. (I. Def. 19.)

Again, because in the triangles *AEB*, *CEB*,
AE is equal to *CE*,
BE to *BE*,
and the angle *AEB* to the angle *CEB*,
the triangles are equal in all respects; (I. Prop. 4.)
therefore *BA* is equal to *BC*.
Therefore the rectangle *ABCD* is a square.
(I. Def. 20.)

Wherefore, *a square ABCD has been inscribed in the given circle ABCD.*

EXERCISES.

1. Inscribe a regular octagon in a given circle.

2. Shew how to cut off the corners of a square so as to leave a regular octagon.

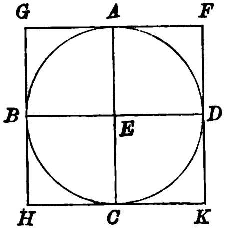
3. Inscribe in a given square, a square to have its sides equal to a given straight line.

PROPOSITION 7.

To describe a square about a given circle.

Let $ABCD$ be the given circle:
it is required to describe a square about it.

CONSTRUCTION. Find E the centre of the circle $ABCD$, (III. Prop. 5.)
and draw two diameters AEC, BED at right angles to one another. (I. Prop. 11.)
Draw GAF, HCK parallel to BD, and GBH, FDK parallel to AC:
the quadrilateral $FGHK$ is a square described as required.



PROOF. Because GF, HK are each parallel to BD,
GF, HK are parallel to each other. (I. Prop. 30.)
Similarly it can be proved that
GH, FK are parallel to each other;
therefore $FGHK$ is a parallelogram.

Again, because $GAEB$ is a parallelogram, (Constr.)
the angle AGB is equal to the angle AEB, (I. Prop. 34.)
which is a right angle; (Constr.)
therefore the parallelogram $FGHK$ is a rectangle. (I. Def. 19.)

Again, because $GBDF$ is a parallelogram,
GF is equal to BD, a diameter of the circle.
Similarly it can be proved that
GH is equal to AC, a diameter of the circle;
therefore GF is equal to GH.
Therefore the rectangle $FGHK$ is a square. (I. Def. 20.)

Again, because *AE* intersects the parallel lines *GF*, *BD*, the angle *GAE* is equal to the alternate angle *AED*, (I. Prop. 29.)
which is a right angle; (Constr.)
therefore *GAF* touches the circle. (III. Prop. 16.)
Similarly it can be proved that *GBH*, *HCK*, *KDF* touch the circle.

Wherefore, *a square FGHK has been described about the given circle ABCD.*

EXERCISES.

1. Describe a regular octagon about a given circle.

2. Prove that the area of a circumscribed square of a circle is double that of an inscribed square.

3. If two circles be such that the same square can be inscribed in one and described about the other, the circles must be concentric.

Is any other condition necessary?

4. If a parallelogram admit of a circle being inscribed in it and another circle being described about it, the parallelogram must be a square.

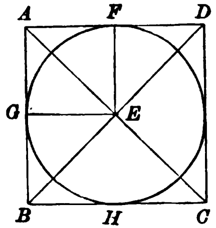
PROPOSITION 8.

To inscribe a circle in a given square.

Let $ABCD$ be the given square:
it is required to inscribe a circle in it.

CONSTRUCTION. Draw AC, BD intersecting in E.

From E draw EF, EG perpendicular to DA, AB two of the sides of the square, (I. Prop. 12.)
and with E as centre and EF or EG as radius describe a circle:
it is a circle inscribed as required.



PROOF. Because CD is equal to AD, (1. Prop. 34, Coroll. 1.)
the angle CAD is equal to the angle ACD. (I. Prop. 5.)
And because AC meets the parallels AB, DC,
the angle BAC is equal to the alternate angle ACD; (I. Prop. 29.)
therefore the angle BAC is equal to the angle DAC.
And because in the triangles GAE, FAE,
the angle GAE is equal to the angle FAE,
and the angle AGE to the angle AFE,
and AE equal to AE,
the triangles are equal in all respects; (I. Prop. 26, Part 2.)
therefore EG is equal to EF,
and therefore the circle described with E as centre and EF or EG as radius passes through the extremity of the other, and touches the two sides DA, AB. (III. Prop. 16.)

Similarly it can be proved that this circle touches each of the sides *BC*, *CD*:

it is therefore inscribed in the square *ABCD*.

Wherefore, *a circle FGH has been inscribed in the given square ABCD.*

EXERCISES.

1. Prove that a circle can be inscribed in any rhombus.

2. Two opposite sides of a convex quadrilateral are together equal to the other two. Shew that a circle can be inscribed in the quadrilateral.

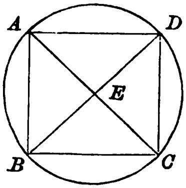
3. *AD*, *BE* are common tangents to two circles *ABC*, *DEC*, that touch each other; shew that a circle may be inscribed in the quadrilateral *ABED*, and a circle may be described about it.

PROPOSITION 9.

To describe a circle about a given square.

Let $ABCD$ be the given square:
it is required to describe a circle about it.

CONSTRUCTION. Draw AC, BD intersecting at E;
and with E as centre and EA, EB, EC or ED as radius describe a circle:
it is a circle described as required.



PROOF. Because in the triangles BAC, DAC,
BA is equal to DA, (I. Prop. 34, Coroll. 1.)
and BC to DC,
and AC to AC,
the triangles are equal in all respects; (I. Prop. 8.)
therefore the angle BAC is equal to the angle DAC,
or the angle BAC is half of the angle BAD.
Similarly it can be proved that
the angle ABD is half of the angle ABC.
But the angle BAD is equal to the angle ABC;
(I. Prop. 29, Coroll. and I. Prop. 10 B.)
therefore the angle BAE is equal to the angle ABE.
Therefore BE is equal to AE. (I. Prop. 6.)
Similarly it can be proved that CE and DE are each of them equal to AE or BE.
Therefore the circle described with E as centre and one of four lines EA, EB, EC, or ED as radius passes through the extremities of the other three, and is described about the square $ABCD$.

Wherefore, *a circle ABCD has been described about the given square ABCD.*

EXERCISES.

1. A point is taken without a square such that the angles subtended at it by three sides of the square are equal: shew that the locus of the point is the circumference of the circle circumscribing the square.

2. Find the locus of a point at which two given sides of a square subtend equal angles.

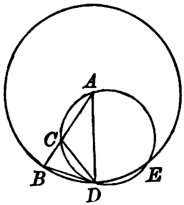
3. If a quadrilateral be capable of having a quadrilateral of minimum perimeter inscribed in it, it must admit of a circle being inscribed in it.

4. $ABCD$ is a quadrilateral inscribed in a circle, and its diagonals, AC, BD intersect at right angles in E; K, L, M, N are the feet of the perpendiculars from E on the sides of the quadrilateral. Shew that $KLMN$ can have circles inscribed in it and described about it.

PROPOSITION 10.

To construct a triangle having each of two angles double of the third angle.

Construction. Take any straight line AB,
and divide it at C, so that the rectangle AB, BC may be equal to the square on AC. (II. Prop. 11.)
With centre A and radius AB describe the circle BDE,
and draw a chord BD equal to AC, (Prop. 1.)
and draw DA:
the triangle ABD is a triangle constructed as required.
Draw DC, and about the triangle ACD describe the circle ACD. (Prop. 5.)



Proof. Because the rectangle AB, BC is equal to the square on AC, and AC is equal to BD,
the rectangle AB, BC is equal to the square on BD.
And because from the point B without the circle ACD,
BCA is drawn cutting it in C and A, and BD is drawn meeting it in D,
and the rectangle AB, BC is equal to the square on BD,
therefore BD touches the circle ACD. (III. Prop. 37.)
And because BD touches the circle ACD, and DC is drawn from the point of contact D,
the angle BDC is equal to the angle DAC. (III. Prop. 32.)
To each of these add the angle CDA;
then the whole angle BDA is equal to the sum of the angles CDA, DAC.
But the angle BCD is equal to the sum of the angles CDA, DAC; (I. Prop. 32.)
therefore the angle BDA is equal to the angle BCD.

But because AD is equal to AB,
the angle ABD is equal to the angle ADB; (I. Prop. 5.)
therefore the angle DBA is equal to the angle BCD.
And because the angle DBC is equal to the angle BCD,
DC is equal to DB. (I. Prop. 6.)
But DB is equal to CA; (Constr.)
therefore CA is equal to CD;
therefore the angle CDA is equal to the angle CAD. (I. Prop. 5.)
Therefore the sum of the angles CAD, CDA is double of the angle CAD.
But the angle BCD is equal to the sum of the angles CAD, CDA; (I. Prop. 32.)
therefore the angle BCD is double of the angle CAD.
And the angle BCD has been proved equal to each of the angles BDA, DBA;
therefore each of the angles BDA, DBA is double of the angle BAD.

Wherefore, *a triangle ABD has been constructed having each of two angles ABD, ADB double of the third angle BAD.*

It will be observed that the smaller angle of the triangle constructed in this proposition is equal to a fifth of two right angles.

EXERCISES.

1. Describe an isosceles triangle having each of the angles at the base one third of the vertical angle.

2. Divide a right angle into five equal parts.

3. In the figure of Proposition 10, if the two circles cut again at E, then DE is equal to DC.

4. In the figure of Proposition 10, the circle ACD is equal to the circle described about the triangle ABD.

5. In the figure of Proposition 10, if AF be the diameter of the smaller circle, DF is equal to a radius of the circle which circumscribes the triangle BCD.

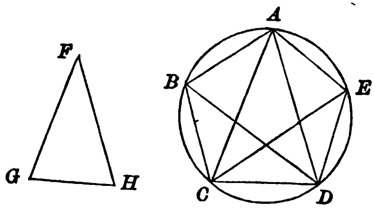
6. If in the figure of Proposition 10, the circles meet again in E, then CE is parallel to BD.

PROPOSITION 11.

To inscribe a regular pentagon in a given circle.

Let $ABCDE$ be the given circle:
it is required to inscribe a regular pentagon in the circle $ABCDE$.

Construction. Construct a triangle FGH having each of the angles at G, H double of the angle at F; (Prop. 10.) in the circle $ABCDE$, inscribe the triangle ACD, equiangular to the triangle FGH, so that the angles CAD, ADC, DCA may be equal to the angles GFH, FHG, HGF respectively. (Prop. 2.)
Bisect the angles ACD, ADC by the straight lines CE, DB, (I. Prop. 9.)
and draw CB, BA, AE, ED:
then $ABCDE$ is a pentagon inscribed as required.



Proof. Because each of the angles ACD, ADC is double of the angle CAD, (Constr.)
and they are bisected by CE, DB,
the five angles ADB, BDC, CAD, DCE, ECA are equal.
But equal angles at the circumference of a circle stand on equal arcs; (III. Prop. 26, Coroll.)
therefore the five arcs AB, BC, CD, DE, EA are equal.
And the chords, by which equal arcs are subtended, are equal; (III. Prop. 29, Coroll.)
therefore the five straight lines AB, BC, CD, DE, EA are equal;
therefore the pentagon $ABCDE$ is equilateral.
Again, the arc ED is equal to the arc BA;
to each of these add the arc AE;

then the whole arc AED is equal to the whole arc BAE.

And the angle AED is contained by the arc AED, and the angle BAE by the arc BAE;

therefore the angle AED is equal to the angle BAE.

(III. Prop. 27, Coroll.)

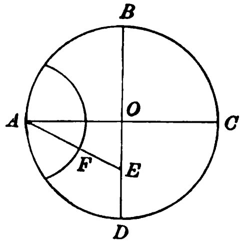
Similarly it can be proved that each of the angles ABC, BCD, CDE is equal to the angle AED or BAE;

therefore the pentagon $ABCDE$ is equiangular.

Therefore $ABCDE$ is a regular pentagon.

Wherefore, *a regular pentagon $ABCDE$ has been inscribed in the given circle $ABCDE$.*

The following is a complete Geometrical construction for inscribing a regular decagon or pentagon in a given circle.



Find O the centre.

Draw two diameters AOC, BOD at right angles to one another.

Bisect OD in E.

Draw AE and cut off EF equal to OE.

Place round the circle ten chords equal to AF.

These chords will be the sides of a regular decagon. Draw the chords joining five alternate vertices of the decagon; they will be the sides of a regular pentagon.

We leave the proof of this as an exercise for the student.

EXERCISES.

1. A regular pentagon is inscribed in a circle, and alternate angular points are joined by straight lines. Prove that these lines will form by their intersections a regular pentagon.

2. If $ABCDE$ be a regular pentagon, $ACEBD$ is a regular star shaped pentagon, each of whose angles is equal to two-fifths of a right angle.

3. $ABCDE$ is a regular pentagon; draw AC and BD, and let BD meet AC at F; shew that AC is equal to the sum of AB and BF.

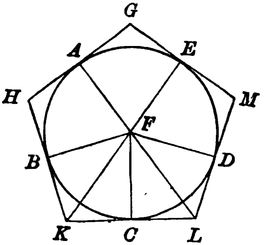
4. If AB, BC, and CD be sides of a regular pentagon, the circle which touches AB and CD at B and C passes through the centre of the circle inscribed in the pentagon.

PROPOSITION 12.

To describe a regular pentagon about a given circle.

Let $ABCDE$ be the given circle:
it is required to describe a regular pentagon about the circle $ABCDE$.

CONSTRUCTION. Let A, B, C, D, E be the angular points of a regular pentagon inscribed in the circle; (Prop. 11.)
so that AB, BC, CD, DE, EA are equal arcs.
Find the centre F; (III. Prop. 5.)
draw FA, FB, FC, FD, FE,
and draw GAH, HBK, KCL, LDM, MEG at right angles to FA, FB, FC, FD, FE respectively; (I. Prop. 11.)
then $GHKLM$ is a pentagon described as required.
Draw FK, FL.



PROOF. Because in the triangles FBK, FCK,
FB is equal to FC, and FK to FK,
and the angle FBK equal to the angle FCK,
each being a right angle, (I. Prop. 10 B.)
and each of the angles FKB, FKC therefore is less than a right angle, (I. Prop. 17.)
the triangles are equal in all respects;
(I. Prop. 26 A, Coroll.)
therefore KB is equal to KC,
and the angle BFK equal to the angle CFK,
and the angle BKF to the angle CKF;
therefore KF bisects each of the angles BFC, BKC.
Similarly it can be proved that LF bisects each of the angles CFD, CLD.

Again, because the arc *BC* is equal to the arc *CD*,
the angle *BFC* is equal to the angle *CFD*.
(III. Prop. 27, Coroll.)
And because the angle *KFC* is half of the angle *BFC*,
and the angle *LFC* is half of the angle *CFD*;
therefore the angle *KFC* is equal to the angle *LFC*.
Now because in the triangles *KFC*, *LFC*,
the angle *KFC* is equal to the angle *LFC*,
and the angle *KCF* to the angle *LCF*,
and *FC* to *FC*,
the triangles are equal in all respects; (I. Prop. 26, Part 1.)
therefore *KC* is equal to *LC*,
and the angle *FKC* equal to the angle *FLC*.
Now it has been proved that *KB* is equal to *KC*,
and that *KL* is double of *KC*;
and it can similarly be proved that *KH* is double of *KB*;
therefore *HK* is equal to *KL*.
Similarly it can be proved that any two consecutive sides of *GHKLM* are equal;
therefore the pentagon *GHKLM* is equilateral.
And because it has been proved that the angles *FKC*, *FLC* are equal,
and that the angle *BKC* is double of the angle *FKC*,
and the angle *CLD* double of the angle *FLC*,
therefore the angle *BKC* is equal to the angle *CLD*.
Similarly it can be proved that any two consecutive angles of *GHKLM* are equal.
Therefore the pentagon *GHKLM* is equiangular.
The pentagon is therefore regular.
And because each side is drawn at right angles to a radius of the circle at its extremity, it touches the circle;
(III. Prop. 16.)
therefore the pentagon is described about the circle *ABCDE*.

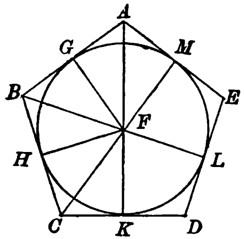
Wherefore, *a regular pentagon GHKLM has been described about the given circle ABCDE.*

PROPOSITION 13.

To inscribe a circle in a given regular pentagon.

Let *ABCDE* be the given regular pentagon:
it is required to inscribe a circle in *ABCDE*.

Construction. Bisect any two consecutive angles of the pentagon *ABC*, *BCD* by *BF*, *CF* (I. Prop. 9.)
meeting at *F*;
draw *FG*, *FH*, *FK*, *FL*, *FM* perpendicular to *AB*, *BC*, *CD*, *DE*, *EA* respectively. (I. Prop. 12.)
With *F* as centre and *FG*, *FH*, *FK*, *FL* or *FM* as radius describe a circle:
it is a circle inscribed as required.
Draw *AF*.



Proof. Because in the triangles *ABF*, *CBF*,
AB is equal to *CB*, (Hypothesis.)
and *BF* to *BF*,
and the angle *ABF* to the angle *CBF*, (Constr.)
the triangles are equal in all respects;
(I. Prop. 4.)
therefore *FA* is equal to *FC*,
and the angle *BAF* is equal to the angle *BCF*.

Again, because the angle *BAE* is equal to the angle *BCD*, (Hypothesis.)
and the angle *BAF* has been proved equal to the angle *BCF*,
and the angle *BCF* is half of the angle *BCD*, (Constr.)
therefore the angle *BAF* is half of the angle *BAE*,
or *AF* bisects the angle *BAE*.

Similarly it can be proved that EF, DF bisect the angles AED, EDC;

therefore the bisectors of all the angles of the pentagon meet in a point.

Again, because in the triangles FCH, FCK,

the angle FHC is equal to the angle FKC,

and the angle FCH to the angle FCK, (Constr.)

and FC to FC,

the triangles are equal in all respects;

(I. Prop. 26, Part 2.)

therefore FH is equal to FK.

Similarly it can be proved that the perpendiculars on any two consecutive sides are equal to one another:

therefore FG, FH, FK, FL, FM are equal, and the circle described with F as centre and one of the five lines FG, FH, FK, FL or FM as radius passes through the extremities of the other four; and because the angles at G, H, K, L, M are right angles, it touches AB, BC, CD, DE, EA. (III. Prop. 16.)

Wherefore, *a circle GHKLM has been inscribed in the given regular pentagon ABCDE.*

EXERCISES.

1. How many conditions are necessary in order that a given pentagon may admit of a circle being inscribed in it?

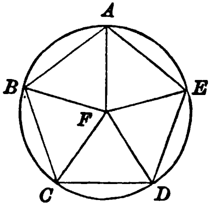
2. Prove that the bisectors of all the angles of any regular polygon meet in a point.

PROPOSITION 14.

To describe a circle about a given regular pentagon.

Let *ABCDE* be the given regular pentagon:
it is required to describe a circle about *ABCDE*.

CONSTRUCTION. Bisect any two consecutive angles of the pentagon *ABC*, *BCD*, by *BF*, *CF* (I. Prop. 9.)
meeting at *F*, and draw *FA*, *FE*, *FD*;
with *F* as centre and *FA*, *FB*, *FC*, *FD* or *FE* as radius describe a circle:
it will be a circle described as required.



PROOF. Because the angle *ABC* is equal to the angle *BCD*, (Hypothesis.)
and the angle *FBC* is half of the angle *ABC*,
and the angle *FCB* is half of the angle *BCD*,
therefore the angle *FBC* is equal to the angle *FCB*;
therefore *FC* is equal to *FB*. (I. Prop. 6.)

Again, because in the triangles *ABF*, *CBF*,
AB is equal to *CB*, (Hypothesis.)
BF to *BF*,
and the angle *ABF* to the angle *CBF*, (Constr.)
the triangles are equal in all respects;
(I. Prop. 4.)
therefore *FA* is equal to *FC*,
and the angle *FAB* to the angle *FCB*.

And because the angle *FAB* is equal to the angle *FCB*,
and the angle *BAE* to the angle *BCD*,
(Hypothesis.)

and the angle *FCB* is half of the angle *BCD*, (Constr.)

therefore the angle *FAB* is half of the angle *BAE*;

and *FA* bisects the angle *BAE*.

Similarly it can be proved that *FD*, *FE* bisect the angles *CDE*, *DEA* respectively,

and that *FD* and *FE* are each of them equal to *FA* or *FC*;

therefore *FA*, *FB*, *FC*, *FD*, *FE* are all equal, and the circle described with *F* as centre and one of the five lines *FA*, *FB*, *FC*, *FD*, *FE* as radius passes through the extremities of the other four, and is described about the pentagon *ABCDE*.

Wherefore, *a circle ABCDE has been described about the given regular pentagon ABCDE.*

EXERCISES.

1. Describe a regular decagon to have five of its vertices coincident with those of a given regular pentagon.

2. How many conditions are necessary in order that a given pentagon may admit of a circle being described about it? State the conditions.

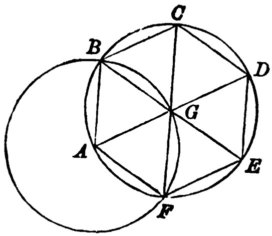
3. Shew how to cut off the corners of a regular pentagon so as to leave a regular decagon.

PROPOSITION 15.

To inscribe a regular hexagon in a given circle.

Let *ABCDEF* be the given circle:
it is required to inscribe a regular hexagon in the circle *ABCDEF*.

Construction. Find *G* the centre of the circle; (III. Prop. 5.)
draw any diameter *AGD*,
and with *A* as centre and *AG* as radius describe the circle *GBF* intersecting the circle *ABCDEF* in *B* and *F*.
Draw *BG*, *FG* and produce them to meet the circle again in *E* and *C*, and draw *AB*, *BC*, *CD*, *DE*, *EF*, *FA*:
then *ABCDEF* is a hexagon inscribed as required.



Proof. Because *G* is the centre of the circle *ABCDEF*,
GB is equal to *GA*.
And because *A* is the centre of the circle *BGF*,
AB is equal to *AG*.
Therefore *AB*, *BG*, *GA* are all equal.
Therefore the angles *AGB*, *BAG*, *GBA* are all equal. (I. Prop. 5, Coroll. 1.)
But the sum of these three angles is equal to two right angles; (I. Prop. 32.)
therefore the angle *AGB* is equal to one-third of two right angles.
Similarly it can be proved that the angle *FGA* is equal to one-third of two right angles.

But the sum of the three angles BGA, AGF, FGE is equal to two right angles; (I. Prop. 13.)
therefore the angle FGE is one-third of two right angles, and therefore the angles BGA, AGF, FGE are equal.
And because opposite vertical angles are equal, (I. Prop. 15.)
the angles opposite to these are equal;
therefore all the angles AGF, FGE, EGD, DGC, CGB, BGA are equal.
Therefore the arcs AF, FE, ED, DC, CB, BA are equal. (III. Prop. 26, Coroll.)
Therefore the chords AF, FE, ED, DC, CB, BA are equal. (III. Prop. 29, Coroll.)
Therefore the hexagon $ABCDEF$ is equilateral.

Again, because the arcs BAF, AFE, FED, EDC, DCB, CBA are equal,
the angles BAF, AFE, FED, EDC, DCB, CBA in those arcs are equal. (III. Prop. 27, Coroll.)
Therefore the hexagon $ABCDEF$ is equiangular:
it is therefore regular, and it is inscribed in the circle $ABCDEF$.

Wherefore, *a regular hexagon $ABCDEF$ has been inscribed in the given circle $ABCDEF$.*

EXERCISES.

1. Inscribe a regular dodecagon in a given circle.

2. If $ABCDEF$ be a regular hexagon, and AC, BD, CE, DF, EA, FB be drawn, they will form another regular hexagon of one third the area.

3. The perimeter of the inscribed equilateral triangle of a circle is three quarters the perimeter of the circumscribed regular hexagon.

4. Six equal circles can be described each touching a given circle and two of the others.

PROPOSITION 16.

To inscribe a regular polygon of fifteen sides in a given circle.

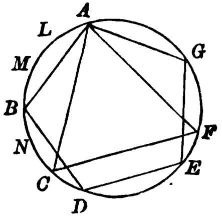
Let $ABCD$ be the given circle:
it is required to inscribe a regular polygon of fifteen sides in the circle $ABCD$.

CONSTRUCTION. Let A, C be two angular points of an equilateral triangle inscribed in the circle, (Prop. 2.)
and let A, B, D be three angular points of a regular pentagon inscribed in the circle. (Prop. 11.

Draw CD,

and place round the circle fifteen chords AL, LM... each equal to CD.

The figure ALM... is a polygon inscribed as required.



PROOF. If the whole circle contain fifteen equal parts, the arc ABC, which is a third of the circle, contains five such parts,
and the arc ABD, which is made up of the arcs AB, BD, each of which is a fifth of the circle, contains six such parts;
therefore the arc CD, which is the difference of the arcs ABD, ABC, consists of one such part;
therefore the arc CD is one-fifteenth of the circle.
And because the arcs AL, LM, ..., which are the shorter arcs cut off by equal chords AL, LM, ..., are equal, (III. Prop. 28, Coroll.)
each of the arcs AL, LM, ..., is one fifteenth of the circle.

Therefore the extremity of the last chord coincides with the point A, and the extremities of the chords which have been placed round the circle exactly divide the circle into fifteen equal parts.

The figure ALM... therefore is equilateral.

And as each of the angles is contained by an arc made up of two of the fifteen equal parts,

all the angles are equal;

(III. Prop. 27, Coroll.)

therefore the figure ALM... is equiangular.

Wherefore, *a regular polygon of fifteen sides has been inscribed in the given circle $ABCD$.*

EXERCISES.

1. Prove that in the figure of Proposition 16 a vertex of the inscribed regular polygon of fifteen sides coincides with each of the vertices of the regular figures used in the construction.

2. Prove that, if all but one of the bisectors of the angles of a polygon meet in a point, they all do so, and a circle can be inscribed in the polygon.

3. Prove that, if all but one of the rectangular bisectors of the sides of a polygon meet in a point, they all do so, and a circle can be described about the polygon.

When a regular polygon of any number of sides is given, we can inscribe a regular polygon of the same number of sides in a given circle, and we can also describe a regular polygon of the same number of sides about a given circle.

Moreover we can always inscribe a circle in a given regular polygon and describe a circle about it.

Methods have now been given for the construction of regular figures of 3, 4, 5, 6, and 15 sides. When any regular polygon is given we can construct a regular polygon of double the number of sides by describing a circle about the polygon and bisecting the smaller arcs subtended by the sides of the given polygon, and so on in succession for each duplication of the number of sides. Thus we see that we can by Euclid's methods construct regular polygons of 3×2^n, 4×2^n, 5×2^n and 15×2^n sides, where n is any positive integer, including zero. It was proved by Gauss* in the year 1801 that by purely geometrical methods those regular polygons can be constructed, the number of whose sides is a prime† number of the form $2^n + 1$. This general law relating to the number of sides includes the case of the triangle ($n=1$) and the pentagon ($n=2$); the next two cases are those of the polygons which have 17 sides ($n=4$) and 257 sides ($n=7$) respectively.

* *Disquisitiones Arithmeticæ* (sectio septima).

† A prime number is an integer which is not divisible without remainder by any integer except itself and unity.

BOOK V.

DEFINITIONS.

DEFINITION 1. *If one magnitude be equal to another magnitude of the same kind repeated twice, thrice or any number of times, the first is said to be* **a multiple** *of the second, and the second is said to be* **an aliquot part** *or* **a measure** *of the first.*

If one magnitude A be equal to m times another magnitude B of the same kind (m being an integer, i.e. a whole number), A is said to be the **mth multiple** of B, and B the **mth part** of A.

If A be any multiple of B and if C be the same multiple of D, then A and C are said to be **equimultiples** of B and D.

The magnitudes treated of in Book V. are not necessarily Geometrical magnitudes: but they are assumed to be such that any magnitude can be supposed to be repeated as often as desired, in other words, that any multiple we please of a magnitude can be taken. They are assumed also to be such that any one taken twice is greater than it is alone; such quantities as those which are called in Algebra either negative or imaginary are excluded from consideration.

The capital letters A, B, C, D &c. will be used to denote magnitudes, and the small letters m, n, p, q &c. to denote whole numbers.

When a magnitude A is spoken of, the letter A is supposed to represent the magnitude itself.

DEFINITION 2. *The relation of one magnitude to another of the same kind with respect to the multiples of the second or of aliquot parts of the second, which the first is greater than, equal to, or less than, is called* **the ratio** *of the first to the second.*

It is difficult to convey a precise idea of "ratio" by a definition. The student will gradually acquire a firmer grasp of the meaning of the term as he proceeds. It is important to bear in mind that the difference between two magnitudes is not their ratio.

Whenever the ratio of one magnitude to another is spoken of, it is necessarily implied, although it may not always be expressly stated, that the two magnitudes are of the same kind.

In the ratio of one magnitude to another, the first is called **the antecedent** *and the second* **the consequent** *of the ratio.*

In the ratio of A to B, A is **the antecedent** and B **the consequent.**

The ratio of a magnitude to an equal magnitude is called **a ratio of equality,** *and is said to be equal to unity;*

the ratio of a magnitude to a less magnitude is called **a ratio of greater inequality,** *and is said to be greater than unity;*

the ratio of a magnitude to a greater magnitude is called **a ratio of less inequality,** *and is said to be less than unity.*

The ratio of one diameter to another diameter of the same circle is a ratio of equality:

the ratio of a diagonal to a side of a square is a ratio of greater inequality:

the ratio of the area of a circle to the area of a square described about the circle is a ratio of less inequality.

DEFINITION 3. *If one magnitude repeated any number of times be greater than, equal to, or less than a second magnitude repeated any other number of times, the ratio of the first magnitude to the second magnitude is said to be greater than, equal to, or less than the ratio of the second number to the first number.*

If A, B be two magnitudes such that m times A is equal to n times B, the ratio of A to B is equal to the ratio of n to m.

Similarly, if m times A be greater than n times B, the ratio of A to B is greater than that of n to m; and if m times A be less than n times B, the ratio of A to B is less than that of n to m.

DEFINITION 4. *When two magnitudes of the same kind are such that some measure of the first is equal to some measure of the second, the two magnitudes are said to be* **commensurable.**

Two magnitudes of the same kind, which are not commensurable, are said to be **incommensurable.**

If A, B, C be three magnitudes of the same kind such that C is the m^{th} part of A and C is the n^{th} part of B, or, in other words, such that A is equal to m times C and B is equal to n times C, then A and B have a common measure C, and therefore are **commensurable.**

If the ratio of A to B be equal to the ratio of one integer to another, say that of n to m, and the ratio of C to D be also equal to that of n to m, the ratio of A to B is equal to that of C to D: and similarly, if the ratio of A to B be equal to that of n to m, and the ratio of C to D be greater or less than that of n to m, the ratio of A to B is less or greater respectively than that of C to D.

A complete method is thus afforded of testing the equality or the inequality of the ratios of pairs of commensurable magnitudes: but the same method is not applicable to incommensurable magnitudes.

Now it will be manifest from what has been said that, if we have four magnitudes A, B, C, D, of which A and B are incommensurable, the ratio of C to D cannot be equal to that of A to B, unless C and D be also incommensurable.

It is possible to find two magnitudes of the same kind that are not commensurable. It can be proved that a diagonal and a side of the same square are such a pair of magnitudes, and also that the circumference and a diameter of the same circle are another such pair of magnitudes. The question arises how the ratios of two pairs of such incommensurable magnitudes are to be compared.

It is easy to prove that a diagonal of a square is greater than once and less than twice a side; these inequalities give a very rough comparison of the lengths of the two lines. It can be proved that 10 times a diagonal is greater than 14 times and less than 15 times a side: these inequalities give a less rough comparison of the lengths. Again, it can be proved that 100 times a diagonal is greater than 141 times and less than 142 times a side: these inequalities give a still less rough comparison of the lengths of the two lines.

These facts are represented by saying that the ratio of a diagonal to a side is greater than the ratio of 1 to 1 and less than that of 2 to 1: greater than the ratio of 14 to 10 and less than that of 15 to 10: greater than the ratio of 141 to 100 and less than that of 142 to 100.

Pairs of ratios of greater and greater numbers might be quoted, between which the ratio of a diagonal to a side always lies: but no two numbers can be found such that the ratio in question is equal to the ratio of the numbers.

In the following definition of the equality of two ratios, the case of incommensurable magnitudes is included as well as that of commensurable magnitudes.

DEFINITION 5. *If four magnitudes be such that, when any equimultiples whatever of the first and the third are taken, and also any equimultiples whatever of the second and the fourth, the multiples of the first and the third are simultaneously either both greater than, or both equal to, or both less than the multiples of the second and the fourth respectively, the ratio of the first magnitude to the second is said to be* **equal** *to the ratio of the third to the fourth.*

When the ratio of the first of four magnitudes to the second is equal to that of the third to the fourth, the magnitudes are said to be **proportionals** *or* **in proportion.**

When four magnitudes are proportionals, the first is said to be to the second as the third to the fourth.

Let A, B, C, D be four magnitudes, of which A and B are of the same kind, and C and D are of the same kind, and let any equimultiples whatever of A and C, say m times A and m times C, be taken, and any equimultiples whatever of B and D, say n times B and n times D; then, if m times A be greater than n times B and also m times C greater than n times D, or else m times A be equal to n times B and also m times C equal to n times D, or else m times A be less than n times B and also m times C less than n times D, for every possible pair of whole numbers m and n, the ratio of A to B is **equal** to the ratio of C to D, and A, B, C, D are **proportionals.**

The fact that four magnitudes A, B, C, D are in proportion is denoted by saying that A has to B the same ratio that C has to D, or that the ratio of A to B is equal to that of C to D, or that A is to B as C to D: it is expressed still more concisely by the notation

$$A : B = C : D.$$

NOTE. It will be observed that when four magnitudes A, B, C, D

are defined in order as proportionals, i.e. A is to B as C to D, they are at the same time defined as proportionals also in the three several orders B, A, D, C; C, D, A, B; and D, C, B, A; that is to say, it follows from the definition that, if any one of the four proportions

$$A : B = C : D, \quad B : A = D : C, \quad C : D = A : B, \quad D : C = B : A$$

exist, the other three exist also.

It follows at once from Definition 5 that if, of A, B, C, D, four magnitudes, A be equal to C and B be equal to D, the ratio of A to B is equal to the ratio of C to D; and further that if, of A, B, C, three magnitudes, A be equal to B, the ratio of A to C is equal to the ratio of B to C, and also the ratio of C to A is equal to the ratio of C to B.

DEFINITION 6. *When four magnitudes are proportionals, the first and the third, the antecedents, are said to be* **homologous** *to one another, and the second and the fourth, the consequents, are also said to be* **homologous** *to one another.*

In the proportion A is to B as C to D, the antecedents A and C are **homologous** to one another and the consequents B and D are **homologous** to one another.

DEFINITION 7. *If it be possible to take equimultiples of the first and the third of four magnitudes and equimultiples of the second and the fourth, such that the multiple of the first is greater than that of the second and the multiple of the third not greater than that of the fourth, the ratio of the first to the second is said to be greater than that of the third to the fourth; and the ratio of the third to the fourth is said to be less than the ratio of the first to the second.*

As an example the ratios of the numbers 2 to 3 and 5 to 8 may be taken.

If the 5^{th} multiples of the first and the third be taken and the 3^{rd} multiples of the second and the fourth, the multiples in order are 10, 9, 25, 24: here 10 is greater than 9 and 25 greater than 24: but equality between the ratios is not thereby established.

If the 11th multiples of the first and the third be taken, and the 7th multiples of the second and the fourth, the multiples in order are 22, 21, 55, 56: here 22 is greater than 21 and 55 not greater than 56: and the fact is established that the ratio of 2 to 3 is greater than that of 5 to 8.

DEFINITION 8. *The ratio of the first of three magnitudes of the same kind to the third is said to be* **compounded** *of the ratio of the first to the second and the ratio of the second to the third.*

The ratio of the first of three magnitudes of the same kind to the third is also said to be **the ratio of the ratio** *of the first magnitude to the second* **to the ratio** *of the third magnitude to the second.*

If A, B, C be three magnitudes of the same kind, the ratio of A to C is **compounded** of the ratio A to B and the ratio B to C.

Further, the ratio of A to C is said to be **the ratio of the ratio** A to B **to the ratio** C to B.

DEFINITION 9. *If the first of a number of magnitudes of the same kind be to the second as the second to the third and as the third to the fourth and so on, the magnitudes are said to be* **in continued proportion.**

If a number of magnitudes be **in continued proportion,** *the ratio of the first to the third is said to be* **duplicate** *of the ratio of the first to the second, and the ratio of the first to the fourth is said to be* **triplicate** *of the ratio of the first to the second.*

If four magnitudes be in proportion, the first and the fourth are called **the extremes** *and the second and the fourth* **the means** *of the proportion.*

If three magnitudes be in continued proportion, the first and the third are called **the extremes** *and the second* **the mean** *of the proportion; also the second is called* **a mean proportional** *between the first and the third, and the third is called* **a third proportional** *to the first and the second.*

PROPOSITION 1. [EUCLID ELEM. PROP. 4.]

*If four magnitudes in order be proportionals and any equimultiples of the antecedents be taken, and any equimultiples of the consequents, the four multiples are proportionals in the same order as the magnitudes**.

Let the magnitudes A, B, C, D be proportionals, and let m, n be two given numbers:

it is required to prove that m times A, n times B, m times C, n times D are proportionals.

CONSTRUCTION. Let p, q be any two numbers, and let m times A be E, n times B be F, m times C be G and n times D be H.

PROOF. Because E, G are equimultiples of A, C,
p times E, p times G are equimultiples of A, C,
no matter what number p may be;
and because F, H are equimultiples of B, D,
q times F, q times H are equimultiples of B, D,
no matter what number q may be;
and because A is to B as C to D,
p times E and p times G
are both greater than, both equal to or both less than q times F and q times H respectively,
for all values of p and q; (Def. 5.)
therefore E is to F as G to H; (Def. 5.)
that is, m times A is to n times B as m times C to n times D.

Wherefore, *if four magnitudes*, &c.

* Algebraically. If $a : b = c : d$, then $ma : nb = mc : nd$.

PROPOSITION 2. [EUCLID ELEM. PROP. 8.]

The greater of two magnitudes has to a third magnitude a greater ratio than the less has; and a third magnitude has to the less of two other magnitudes a greater ratio than it has to the greater.*

Let A, B, C be three magnitudes of the same kind, of which A is greater than B:
it is required to prove that A has to C a greater ratio than B has to C, and that C has to B a greater ratio than C has to A.

CONSTRUCTION. Let the excess of A over B be D;
take the m^{th} equimultiples of B, D, such that each is greater than C, and of the multiples of C let the p^{th} multiple be the first which is greater than m times B, and let n be the number next less than p.

PROOF. Because m times B is not less than n times C,
and m times D is greater than C; (Constr.)
therefore the sum of m times B and m times D is greater than the sum of n times C and C,
that is, m times A is greater than p times C;
and m times B is less than p times C; (Constr.)
therefore A has to C a greater ratio than B has to C. (Def. 7.)

Next, because p times C is less than m times A,
and p times C is greater than m times B,
therefore C has to B a greater ratio than C has to A. (Def. 7.)

Wherefore, *the greater*, &c.

* Algebraically. If $a>b$, then $a : c>b : c$ and $c : b>c : a$.

PROPOSITION 3. [EUCLID ELEM. PROP. 9.]

*If the ratio of the first of three magnitudes to the third be equal to the ratio of the second to the third, the first magnitude is equal to the second**.

Let A, B, C be three magnitudes of the same kind such that A is to C as B is to C:
it is required to prove that A is equal to B.

PROOF. Because, any magnitude greater than B has to C a greater ratio than B has to C, (Prop. 2.)
and A has to C the same ratio as B to C,
A cannot be greater than B.

Again, because any magnitude less than B has to C a less ratio than B to C, (Prop. 2.)
and A has to C the same ratio as B to C,
A cannot be less than B.
Therefore A must be equal to B.

Wherefore, *if the ratio of the first*, &c.

* Algebraically. If $a : c = b : c$, then $a = b$.

PROPOSITION 4. PART 1. [EUCLID ELEM. PROP. 10.]

If the ratio of the first of three magnitudes to the third be greater than the ratio of the second to the third, the first magnitude is greater than the second.*

Let A, B, C be three magnitudes of the same kind, such that A has to C a greater ratio than B to C:
it is required to prove that A is greater than B.

PROOF. Because the ratio of A to C is greater than that of B to C,
there are some equimultiples, say the m^{th} multiples, of A, B and some multiple, say the n^{th} multiple, of C, such that m times A is greater than n times C,
and m times B not greater than n times C; (Def. 7.)
therefore there is some number m such that m times A is greater than m times B;
therefore A is greater than B.

Wherefore, *if the ratio of the first*, &c.

PROPOSITION 4. PART 2. [EUCLID ELEM. PROP. 10.]

If the ratio of the first of three magnitudes to the second be greater than the ratio of the first to the third, the second magnitude is less than the third†.

Let A, B, C be three magnitudes of the same kind, such that A has to B a greater ratio than A to C:
it is required to prove that B is less than C.

PROOF. Because the ratio of A to B is greater than that of A to C, there is some multiple, say the m^{th} multiple, of A, and there are some equimultiples, say the n^{th} multiples, of B, C, such that m times A is greater than n times B,
and m times A not greater than n times C; (Def. 7.)
therefore there is some number n such that n times C is greater than n times B;
therefore C is greater than B.

Wherefore, *if the ratio of the first*, &c.

* Algebraically. If $a : c > b : c$, then $a > b$.
† Algebraically. If $a : b > a : c$, then $b < c$.

PROPOSITION 5. [EUCLID ELEM. PROP. 11.]

*Ratios, which are equal to the same ratio, are equal to one another**.

Let A, B, C, D, E, F be six magnitudes, such that

A is to B as C to D,

and also E is to F as C to D:

it is required to prove that

A is to B as E to F.

PROOF. Because A is to B as C to D,

m times A and m times C

are both greater than, both equal to or both less than n times B and n times D respectively,

for all values of m and n; (Def. 5.)

and because E is to F as C to D,

m times E and m times C

are both greater than, both equal to, or both less than n times F and n times D respectively,

for all values of m and n;

therefore m times A and m times E

are both greater than, both equal to or both less than n times B and n times F respectively,

for all values of m and n;

therefore A is to B as E to F. (Def. 5.)

Wherefore, *ratios, which are equal*, &c.

* Algebraically. If $a : b = c : d$, and $e : f = c : d$,
then $a : b = e : f$.

PROPOSITION 6. [Euclid Elem. Prop. 12.]

If any number of ratios be equal, each ratio is equal to the ratio of the sum of the antecedents to the sum of the consequents.*

Let A, B, C, D, E, F be any number of magnitudes of the same kind, such that the ratios of A to B, C to D, E to F are equal:

it is required to prove that

A is to B as the sum of A, C, E to the sum of B, D, F.

Construction. Take any equimultiples, say the m^{th} multiples, of A, C, E, and any equimultiples, say the n^{th} multiples, of B, D, F.

Proof. Because A is to B as C to D, and also as E to F,

therefore m times A, m times C and m times E are simultaneously all greater than, all equal to or all less than n times B, n times D and n times F respectively,

for all values of m and n; (Def. 5.)

therefore m times A and m times the sum of A, C, E are simultaneously all greater than, all equal to or all less than n times B and n times the sum of B, D, F respectively, for all values of m and n.

Therefore

A is to B as the sum of A, C, E to the sum of B, D, F. (Def. 5.)

Wherefore, *if any number of ratios, &c.*

Corollary. *The ratio of two magnitudes is equal to the ratio of any two equimultiples of them†.*

* Algebraically. If $a : b = c : d = e : f$,
then $a : b = a+c+e : b+d+f$.

† Algebraically. $a : b = ma : mb$.

PROPOSITION 7. [EUCLID ELEM. PROP. 13.]

*If the first of three ratios be equal to the second and the second greater than the third, the first is greater than the third**.

Let A, B, C, D, E, F be six magnitudes, such that the ratio of A to B is equal to that of C to D, and the ratio of C to D is greater than that of E to F:
it is required to prove that the ratio of A to B is greater than that of E to F.

PROOF. Because the ratio of C to D is greater than that of E to F, it is possible to find some equimultiples, say the m^{th} multiples, of C and E, and some equimultiples, say the n^{th} multiples, of D and F, such that

m times C is greater than n times D
and m times E not greater than n times F. (Def. 7.)

Again, because A is to B as C to D,
m times A and m times C
are simultaneously both greater than, both equal to, or both less than n times B and n times D respectively,
for all values of m and n. (Def. 5.)

Therefore for some values of m and n
m times A is greater than n times B
and m times E not greater than n times F.

Therefore the ratio of A to B is greater than that of E to F. (Def. 7.)

Wherefore, *if the first of three ratios*, &c.

* Algebraically. If $a : b = c : d$, and $c : d > e : f$,
then $a : b > e : f$.

PROPOSITION 8. [EUCLID ELEM. PROP. 14.]

If the first of four proportionals of the same kind be greater than the third, the second is greater than the fourth; if the first be equal to the second, the third is equal to the fourth; if the first be less than the second, the third is less than the fourth.*

Let A, B, C, D be four magnitudes of the same kind such that A is to B as C to D:

it is required to prove that, if A be greater than C, B is greater than D, and, if A be equal to C, B is equal to D, and, if A be less than C, B is less than D.

PROOF. First. Let A be greater than C. Because A, B, C are three magnitudes and A is greater than C,

the ratio of A to B is greater than that of C to B; (Prop. 2.)

but the ratio of A to B is equal to that of C to D;

therefore the ratio of C to D is greater than that of C to B; (Prop. 7.)

therefore B is greater than D. (Prop. 4, Part 2.)

Next. Let A be equal to C.

Because A is to B as C to D,

B is to A as D to C; (Def. 5. Note.)

and A is equal to C;

therefore B is to C as D to C;

therefore B is equal to D. (Prop. 3.)

Lastly. Let A be less than C.

Because A is to B as C to D,

C is to D as A to B; (Def. 5. Note.)

therefore, by the first case,

if C be greater than A, D is greater than B,

that is, if A be less than C, B is less than D.

Wherefore, *if the first*, &c.

* Algebraically. If $a : b = c : d$,

then $a >=< c$ according as $b >=< d$.

PROPOSITION 9. [EUCLID ELEM. PROP. 16.]

*If the first of four magnitudes of the same kind be to the second as the third to the fourth, then also the first is to the third as the second to the fourth**.

Let A, B, C, D be four magnitudes of the same kind such that A is to B as C to D:

it is required to prove that A is to C as B to D.

CONSTRUCTION. Take any equimultiples, say the m^{th} multiples, of A, B, and any equimultiples, say the n^{th} multiples, of C, D.

PROOF. Because m times A is to m times B as A to B, and because n times C is to n times D as C to D, (Prop. 6. Coroll.)

and A is to B as C to D; (Hypothesis.)

therefore m times A is to m times B as n times C to n times D. (Prop. 5.)

Therefore m times A and m times B are both greater than, both equal to or both less than n times C and n times D respectively, for all values of m and n; (Prop. 8.)

therefore A is to C as B to D. (Def. 5.)

Wherefore, *if the first*, &c.

* Algebraically. If $a : b = c : d$, then $a : c = b : d$.

PROPOSITION 10. [EUCLID ELEM. PROP. 17.]

If the sum of the first and the second of four magnitudes be to the second as the sum of the third and the fourth to the fourth, the first is to the second as the third to the fourth.*

Let A, B, C, D be four magnitudes, A and B being of the same kind and C and D of the same kind, such that
the sum of A and B is to B as the sum of C and D to D:
it is required to prove that
A is to B as C to D.

CONSTRUCTION. Take any equimultiples, say the m^{th} multiples, of A, B, C, D and any equimultiples, say the n^{th} multiples, of B, D;
then the sums of m times B and n times B and of m times D and n times D are equimultiples of B and D respectively.

PROOF. Because the sum of A and B is to B as the sum of C and D to D, (Hypothesis.)
therefore m times the sum of A and B
and m times the sum of C and D are simultaneously both greater than, both equal to or both less than
the sum of m and n times B
and the sum of m and n times D respectively,
for all values of m and n: (Def. 5.)
therefore m times A and m times C
are simultaneously both greater than, both equal to or both less than n times B and n times D respectively,
for all values of m and n;
therefore A is to B as C to D. (Def. 5.)

Wherefore, *if the sum*, &c.

* Algebraically. If $a+b : b=c+d : d$, then $a : b=c : d$.

PROPOSITION 11. [Euclid Elem. Prop. 18.]

*If the first of four magnitudes be to the second as the third to the fourth, then the sum of the first and the second is to the second as the sum of the third and the fourth to the fourth**.

Let A, B, C, D be four magnitudes, A and B being of the same kind, and C and D of the same kind, such that A is to B as C to D:

it is required to prove that

the sum of A and B is to B as the sum of C and D to D.

Construction. Take any equimultiples, say the m^{th} multiples, of A, C and any equimultiples, say the n^{th} multiples, of B, D.

Proof. Because A is to B as C to D, (Hypothesis.)

therefore m times A and m times C

are simultaneously both greater than, both equal to or both less than n times B and n times D respectively,

for all values of m and n; (Def. 5.)

therefore m times the sum of A and B

and m times the sum of C and D

are simultaneously both greater than, both equal to or both less than the sum of m and n times B

and the sum of m and n times D respectively,

for all values of m and n.

And it is manifest that

m times the sum of A and B

and m times the sum of C and D

are simultaneously both greater than

p times B and p times D respectively,

for all values of m and p, where m is not less than p.

Therefore m times the sum of A and B

and m times the sum of C and D

are simultaneously both greater than, both equal to or both less than p times B and p times D respectively,

for all values of m and p;

therefore the sum of A and B is to B as the sum of C and D to D. (Def. 5.)

Wherefore, *if the first*, &c.

* Algebraically. If $a : b = c : d$, then $a+b : b = c+d : d$.

PROPOSITION 12. [EUCLID ELEM. PROP. 19.]

*If the sum of the first and the second of four magnitudes be to the sum of the third and the fourth as the second to the fourth, the first is to the second as the third to the fourth**.

Let A, B, C, D be four magnitudes of the same kind, such that

the sum of A and B is to the sum of C and D as B to D:
it is required to prove that A is to B as C to D.

PROOF. Because
the sum of A and B is to the sum of C and D as B to D,
the sum of A and B is to B as the sum of C and D to D. (Prop. 9.)
Therefore A is to B as C is to D. (Prop. 10.)

Wherefore, *if the sum*, &c.

* Algebraically. If $a+b : c+d = b : d$, then $a : b = c : d$.

PROPOSITION 13. [Euclid Elem. Prop. 20.]

If the first of six magnitudes be to the second as the fourth to the fifth, and the second be to the third as the fifth to the sixth, then the first and the fourth are both greater than, both equal to, or both less than the third and the sixth respectively.*

Let A, B, C, D, E, F be six magnitudes, A, B, C being of the same kind and D, E, F of the same kind, such that

A is to B as D to E, and B is to C as E to F:

it is required to prove that A and D are both greater than, both equal to or both less than C and F respectively.

Proof. First, let A be greater than C.

Because A is to B as D to E, (Hypothesis.)
and the ratio of A to B is greater than that of C to B, (Prop. 2.)
the ratio of D to E is greater than that of C to B; (Prop. 7.)
and because C is to B as F to E; (Def. 5, Note.)
the ratio of D to E is greater than that of F to E; (Prop. 7.)
therefore D is greater than F. (Prop. 4.)

Secondly, because the magnitudes are proportionals when taken in the orders A, B, D, E; B, C, E, F, they are also proportionals when taken in the orders D, E, A, B; E, F, B, C; (Def. 5, Note.)
therefore by the first case,

if D be greater than F, A is greater than C.

Lastly. The magnitudes are also proportionals when taken in the orders C, B, F, E; B, A, E, D; (Def. 5, Note.)
therefore by the first and second cases,

if C be greater than A, F is greater than D,
and if F be greater than D, C is greater than A;

therefore A and D are both greater than, both equal to or both less than C and F respectively.

Wherefore, *if the first*, &c.

* Algebraically. If $a : b = d : e$ and $b : c = e : f$,
then $a >=< c$ according as $d >=< f$.

PROPOSITION 14. [EUCLID ELEM. PROP. 22.]

If the first of six magnitudes be to the second as the fourth to the fifth, and the second be to the third as the fifth to the sixth, then the first is to the third as the fourth to the sixth.*

Let A, B, C, D, E, F be six magnitudes, A, B, C being of the same kind and D, E, F of the same kind, such that
A is to B as D to E, and B is to C as E to F:
it is required to prove that A is to C as D to F.

CONSTRUCTION. Take any equimultiples, say the m^{th} multiples, of A, D,
and any equimultiples, say the n^{th} multiples, of B, E,
and any equimultiples, say the p^{th} multiples, of C, F.

PROOF. Because A is to B as D to E,
and B is to C as E to F;
therefore m times A is to n times B as m times D to n times E,
and n times B is to p times C as n times E to p times F. (Prop. 1.)

Therefore m times A and m times D
are both greater than, both equal to or both less than p times C and p times F respectively,
for all values of m and p. (Prop. 13.)
Therefore A is to C as D to F. (Def. 5.)

Wherefore, *if the first*, &c.

COROLLARY. *Ratios which are duplicate of equal ratios are equal†.*

* Algebraically. If $a : b = d : e$ and $b : c = e : f$, then $a : c = d : f$.

† Algebraically. If $a : b = b : c$ and $d : e = e : f$ and $a : b = d : e$,
then $$a : c = d : f.$$

PROPOSITION 15.

*If the first of six magnitudes have to the second a greater ratio than the fourth to the fifth, and the second have to the third a greater ratio than the fifth to the sixth, then the first has to the third a greater ratio than the fourth to the sixth**.

Let A, B, C, D, E, F be six magnitudes, A, B, C being of the same kind, and D, E, F of the same kind, such that

A has to B a greater ratio than D to E,

and B has to C a greater ratio than E to F:

it is required to prove that

A has to C a greater ratio than D to F.

Construction. Because the ratio of A to B is greater than that of D to E, it is possible to find some equimultiples, say the m^{th} multiples, of A and D, and some equimultiples, say the n^{th} multiples, of B and E, such that

m times A is greater than n times B,

and m times D not greater than n times E: (Def. 7.)

and because the ratio of B to C is greater than that of E to F, it is possible to find some equimultiples, say the p^{th} multiples, of B and E, and some equimultiples, say the q^{th} multiples, of C and F, such that

p times B is greater than q times C,

and p times E not greater than q times F. (Def. 7.)

Let p times m be r and n times q be s,

and let n times B be H and p times B be K.

Proof. Because m times A is greater than n times B,

and p times m is r, and n times B is H,

therefore r times A is greater than p times H;

and because p times B is greater than q times C,

and p times B is K and n times q is s,

therefore n times K is greater than s times C;

and because n times B is H, and p times B is K,

p times H is equal to n times K;

therefore r times A is greater than s times C.

Similarly it can be proved that

r times D is not greater than s times F;

therefore A has to C a greater ratio than D to F. (Def. 7.)

Wherefore, *if the first*, &c.

* Algebraically. If $a : b > d : e$ and $b : c > e : f$, then $a : c > d : f$.

PROPOSITION 16.

Ratios, of which equal ratios are duplicate, are equal.*

Let A, B, C, D, E, F be six magnitudes, A, B, C being of the same kind, and D, E, F of the same kind, such that

A is to B as B to C,
and D is to E as E to F,
and also A is to C as D to F:

it is required to prove that

A is to B as D to E.

PROOF. If the ratio of A to B were greater than that of D to E,

then also, since A is to B as B to C,
and D is to E as E to F,

the ratio of B to C would be greater than that of E to F; (Prop. 7.)

therefore the ratio of A to C would be greater than that of D to F. (Prop. 15.)

Similarly, if the ratio of A to B were less than D to E, then the ratio of A to C would be less than that of D to F;

therefore A is to B as D to E.

Wherefore, *ratios, of which* &c.

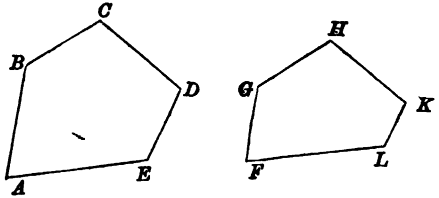
* Algebraically. If $a : c = d : f$ and $a : b = b : c$ and $d : e = e : f$, then $a : b = d : e$, negative quantities being excluded.

BOOK VI.

DEFINITIONS.

It is often convenient to speak of closed rectilineal figures as a class. The wording of definition 15 of Book I. (page 11) implies that the term **polygon** does not include a triangle or a quadrilateral. This restriction for the future will not be maintained, and any closed rectilineal figure, no matter what the number of its sides may be, will be included under the term polygon.

DEFINITION 1. *When the angles of one polygon taken in order are equal to the angles of another taken in order, the two* **polygons** *are said to be* **equiangular to one another.**



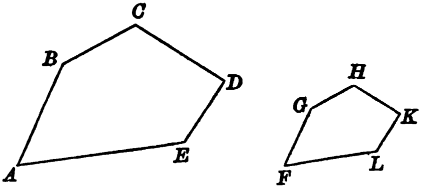
The polygons $ABCDE$, $FGHKL$ are equiangular to one another, if the angles at A, B, C, D, E be equal to the angles at F, G, H, K, L respectively.

Pairs of vertices A, F; B, G; &c., at which the angles are equal, are **corresponding vertices**: and pairs of sides AB, FG; BC, GH; &c. joining corresponding pairs of vertices are **corresponding sides**.

In this definition there is one more condition of equality than is necessary. If $n-1$ of the angles of a polygon of n sides be equal to $n-1$ of the angles of another polygon of n sides, the remaining angles must be equal. (See I. Prop. 32, Coroll.)

DEFINITION 2. *When the ratio of a side of one of two polygons, which are equiangular to one another, to the corresponding side of the other is the same for all pairs of corresponding sides, the polygons are said to be* **similar to one another.**

The polygons *ABCDE*, *FGHKL* are **similar to one another**, if the angles at *A*, *B*, *C*, *D*, *E* be equal to the angles at *F*, *G*, *H*, *K*, *L*



respectively, and if also all the ratios of *AB* to *FG*, *BC* to *GH*, *CD* to *HK*, *DE* to *KL*, *EA* to *LF* be equal to one another.

It will be seen hereafter that there are in this definition three more conditions of equality than are necessary. One unnecessary condition is contained in the statement that the polygons are equiangular to one another. Other two unnecessary conditions are contained in the statement that *all* the ratios are equal. For it can be proved that in general, if in two *equiangular* polygons all but two of the ratios of corresponding sides be equal, all the ratios are equal.

DEFINITION 3. *If in each of two given finite straight lines a point be taken such that the segments of the first line are in the same ratio as the segments of the second line, the two lines are said to be* **cut proportionally** *by the points.*

In the diagram (figure 1) the points *P*, *Q* cut the straight lines *AB*, *CD* proportionally, if *AP* be to *PB* as *CQ* to *QD*.

This definition is extended also to the case, where the points *P* and *Q* are in the lines *AB*, *CD* produced (figure 2). It must however be noticed that both points *P*, *Q* must be in the lines themselves, or both points in the lines produced: otherwise the lines are not said to be cut proportionally.

In figure 1 the points *P*, *Q* are said to cut the lines *AB*, *CD* **internally**; in figure 2 the points *P*, *Q* are said to cut the lines *AB*, *CD* **externally**.

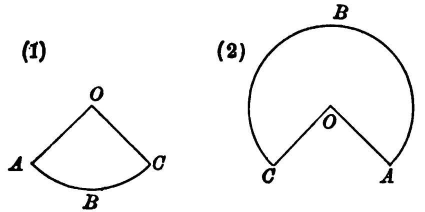
DEFINITION 4. *In some cases, where one side of a triangle is specially distinguished from the other two sides, that side is called* **the base of the triangle** *and the perpendicular* upon that side from the opposite vertex is called* **the altitude of the triangle.**

Similarly, *one side of a parallelogram is sometimes called the base and the perpendicular distance between it and the opposite side the altitude of the parallelogram.*

DEFINITION 5. *When a straight line is divided into two parts, so that the whole is to one part as that part to the other part, the line is said to be* **divided in extreme and mean ratio.**

DEFINITION 6. *The figure formed of an arc of a circle and the radii drawn to its extremities is called* **a sector of the circle.**

The angle between the radii, which is subtended by the arc, is called **the angle of the sector.**



If O be the centre of the circle, of which the arc ABC is a part, and OA, OC be radii, the figure $OABC$ is a sector.

In figure 1 the angle of the sector is less than two right angles, in figure 2 the angle of the sector is greater than two right angles.

DEFINITION 7. *Points lying on a straight line are said to be* **collinear.** *A set of such points is called* **a range.**

Straight lines passing through a point are said to be **concurrent.** *A set of such lines is called* **a pencil.** *The lines are called* **the rays** *of the pencil and the point is called* **the vertex** *of the pencil.*

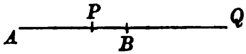
A set of points $ABCD$... lying on a straight line is called **the range ABCD....**

* (See I. Def. 11, p. 9.)

A set of straight lines drawn from a point O to a series of points A, B, C, D... is called **the pencil O (ABCD...)**.

DEFINITION 8. *Four points on a straight line, such that one pair divide the straight line joining the other pair internally and externally in the same ratio, are called* **a harmonic range.**

In the diagram, if AP be to PB as AQ to QB, then A, P, B, Q is a harmonic range.



Because AP is to PB as AQ to QB, therefore QB is to BP as QA to AP (V. Def. 5, Note, and V. Prop. 9), that is, the points B, A divide the distance QP internally and externally in the same ratio.

The two points which form either pair of a harmonic range are said to be **conjugate to one another.**

The points A, B and P, Q are two pairs of **conjugate points.**

DEFINITION 9. *A pencil of four rays passing through the four points of a harmonic range is called* **a harmonic pencil.**

Two rays, which pass through a pair of conjugate points of a harmonic range, are called **conjugate rays** *of the pencil.*

DEFINITION 10. *Four points on a straight line, such that one pair divide the straight line joining the other pair internally and externally in different ratios, are called* **an anharmonic range.**

The ratio of the ratio of internal division to the ratio of external division is called the ratio of the anharmonic range.

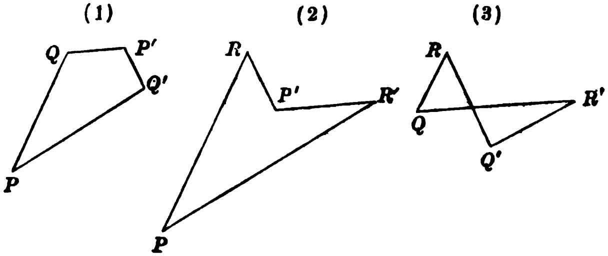
If two anharmonic ranges have equal ratios, they are called **like anharmonic ranges.**

DEFINITION 11. *A pencil of four rays passing through the four points of an anharmonic range is called* **an anharmonic pencil.**

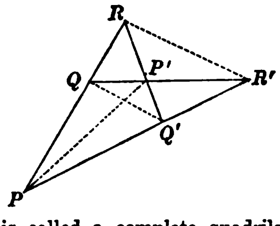
The ratio of the range is also said to be **the ratio of the pencil.**

If two pencils have equal ratios, they are called **like anharmonic pencils.**

Quadrilaterals are often divided into three classes (1) convex, (2) re-entrant, (3) cross, the natures of which appear from the adjoining diagrams.



If the sides of a quadrilateral be produced both ways, the character of the complete figure which is so formed is independent of the class to which the quadrilateral belongs, as is evident from the adjoining diagram.



Such a figure is called a complete quadrilateral, of which the following is a definition.

DEFINITION 12. *The figure formed by four infinite straight lines is called* **a complete quadrilateral.**

The straight line joining the intersection of one pair of lines to the intersection of the other pair is called **a diagonal.**

There are *three* such diagonals. In the last diagram PP', QQ', RR' are **diagonals** of the complete quadrilateral there represented.

PROPOSITION 1.

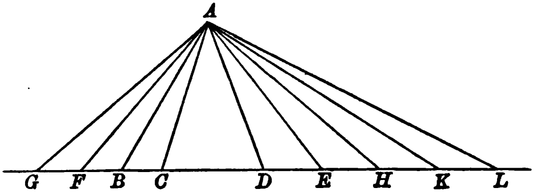
The ratio of two triangles of the same altitude is equal to the ratio of their bases.

Let the triangles ABC, ADE be two triangles of the same altitude, that is, having a common vertex A and their bases BC, DE in a straight line:

it is required to prove that the triangle ABC is to the triangle ADE as BC to DE.

CONSTRUCTION. In CB produced, take any number of straight lines BF, FG each equal to BC, and in DE produced, any number EH, HK, KL each equal to DE;

and draw AF, AG, AH, AK, AL.



PROOF. Because BC, FB, GF are equal,
the triangles ABC, AFB, AGF are equal. (I. Prop. 38, Coroll.)
Therefore the triangle AGC is the same multiple of the triangle ABC, that GC is of BC.

Similarly it can be proved that
the triangle ADL is the same multiple of the triangle ADE, that DL is of DE.

Again, if GC be equal to DL,
the triangle AGC is equal to the triangle ADL,
(I. Prop. 38, Coroll.)
and if GC be greater or less than DL, the triangle AGC is greater or less respectively than the triangle ADL.

Therefore since of the four magnitudes the triangles ABC, ADE and the lines BC, DE, the triangle AGC and the line GC are any equimultiples whatever of the first and the third, and the triangle ADL and the line DL are any equimultiples whatever of the second and the fourth, and it has been proved that

the triangle AGC and GC are both greater than, both equal to or both less than the triangle ADL and DL respectively;

therefore

the triangle ABC is to the triangle ADE as BC to DE. (V. Def. 5.)

Wherefore, *the ratio of two triangles*, &c.

COROLLARY 1. *The ratio of two triangles of equal altitudes is equal to the ratio of their bases.*

COROLLARY 2. *The ratio of two triangles of equal bases is equal to the ratio of their altitudes.*

COROLLARY 3. *The ratio of two parallelograms of equal altitudes is equal to the ratio of their bases.*

Each parallelogram is double of the triangle on the same base and of the same altitude. The ratio of the parallelograms therefore is equal to the ratio of the triangles. (V. Prop. 6, Coroll.)

EXERCISES.

1. The diagonals of a convex quadrilateral, two of whose sides are parallel and one of them double of the other, cut one another at a point of trisection.

2. The sum of the perpendiculars on the two sides of an isosceles triangle from any point of the base is constant.

3. If straight lines AO, BO, CO be drawn from the vertices of a triangle ABC, and AO produced cut BC in D, the triangles AOB, AOC have the same ratio as BD, DC.

4. If in the sides BC, CA of a triangle points D, E be taken, such that BD is twice DC, and CE twice EA, and the straight lines AD, BE intersect in O, then the areas of the triangles EOA, AOB, BOD, ABC are in the ratios of the numbers 1, 6, 8, 21.

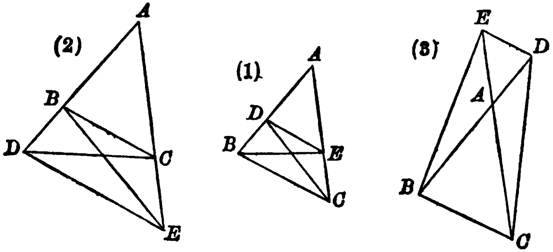
PROPOSITION 2. PART 1.

If a straight line be parallel to one side of a triangle, it cuts the other sides proportionally.

Let the straight line DE be parallel to the side BC of the triangle ABC, and cut the sides AB, AC or these sides produced in D, E respectively:

it is required to prove that BD is to DA as CE to EA.

CONSTRUCTION. Draw BE, CD.



PROOF. Because the two triangles BDE, CDE have the side DE common, and BC is parallel to DE,

the triangles BDE, CDE are equal. (I. Prop. 37.)

Therefore the triangle BDE is to the triangle ADE as the triangle CDE to the triangle ADE. (V. Def. 5, Note.)

But the triangle BDE is to the triangle ADE as BD to DA; (Prop. 1.)

and the triangle CDE is to the triangle ADE as CE to EA; (Prop. 1.)

therefore BD is to DA as CE to EA. (V. Prop. 5.)

Wherefore, *if a straight line* &c.

COROLLARY. Because BC is parallel to DE a side of the triangle ADE, it follows that

DB is to BA as EC to CA;

therefore also AB is to AD as AC to AE.

(V. Props. 10, and 11.)

EXERCISES.

1. A straight line drawn parallel to *BC*, one of the sides of a triangle *ABC*, meets *AB* at *D* and *AC* at *E*; if *BE* and *CD* meet at *F*, then the triangle *ADF* is equal to the triangle *AEF*.

2. If in a triangle *ABC* a straight line parallel to *BC* meet *AB* at *D* and *AC* at *E*, and if *BE* and *CD* meet at *F*: then *AF* produced if necessary will bisect *BC* and *DE*.

3. Through *D*, any point in the base of a triangle *ABC*, straight lines *DE*, *DF* are drawn parallel to the sides *AB*, *AC*, and meet the sides at *E*, *F*: shew that the triangle *AEF* is a mean proportional between the triangles *FBD*, *EDC*.

4. If two sides of a quadrilateral be parallel, any straight line drawn parallel to them will cut the other sides proportionally.

5. If a straight line *EF*, drawn parallel to the diagonal *AC* of a parallelogram *ABCD*, meet *AD*, *DC*, or those sides produced, in *E* and *F* respectively, then the triangle *ABE* is equal in area to the triangle *BCF*.

6. *ABC* is a triangle, and through *D*, a point in *AB*, *DE* is drawn parallel to *BC* meeting *AC* in *E*. Through *C* a line *CF* is drawn parallel to *BE*, meeting *AB* produced in *F*. Prove that *AB* is a mean proportional between *AD* and *AF*.

7. Through a given point within a given angle draw a straight line such that the segments intercepted between the point and the lines which form the angle may have to one another a given ratio.

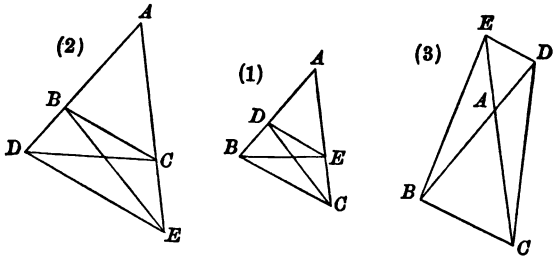
8. Find a point *D* in the side *AB* of a triangle *ABC* such that the square on *CD* is in a given ratio to the rectangle *AD*, *DB*.

PROPOSITION 2. PART 2.

If a straight line cut two sides of a triangle proportionally, it is parallel to the third side.

Let the straight line *DE* cut the sides *AB*, *AC* of the triangle *ABC*, or these sides produced, proportionally in *D*, *E* respectively, so that *BD* is to *DA* as *CE* to *EA*:
it is required to prove that *DE* is parallel to *BC*.

CONSTRUCTION. Draw *BE*, *CD*.



PROOF. Because *BD* is to *DA* as *CE* to *EA*, (Hypothesis.)
and as *BD* to *DA* so is the triangle *BDE* to the triangle *ADE*, (Prop. 1.)
and as *CE* to *EA* so is the triangle *CDE* to the triangle *ADE*; (Prop. 1.)
therefore the triangle *BDE* is to the triangle *ADE* as the triangle *CDE* to the triangle *ADE*; (V. Prop. 5.)
i.e. the triangles *BDE*, *CDE* have the same ratio to the triangle *ADE*;
therefore the triangles *BDE*, *CDE* are equal. (V. Prop. 3.)
But these triangles have a common side *DE* and lie on the same side of it;
therefore *BC* is parallel to *DE*. (I. Prop. 39.)

Wherefore, *if a straight line &c.*

COROLLARY. If AB be to AD as AC to AE,
it follows that BD is to DA as CE to EA
(V. Props. 10 and 11.)
and therefore BC is parallel to AD.

EXERCISES.

1. Prove that there is only one point which divides a given straight line internally in a given ratio, and only one point which divides a given straight line externally in a given ratio.

2. If DEF be a triangle inscribed in a triangle ABC and have its sides parallel to those of ABC, then D, E, F must be the middle points of BC, CA, AB.

3. From a point E in the common base of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G: shew that FG is parallel to CD.

4. If two given distances PQ, RS be measured off on two fixed parallel straight lines, then the locus of the intersection of each of the pairs PS, QR and PR, QS is a parallel straight line.

5. On three straight lines OAP, OBQ, OCR the points are chosen so that AB, PQ are parallel and BC, QR are parallel; prove that AC, PR also are parallel.

6. If two opposite sides AB, DC of a quadrilateral $ABCD$ be parallel, any straight line PQ which cuts AD, BC proportionally must be parallel to AB and DC.

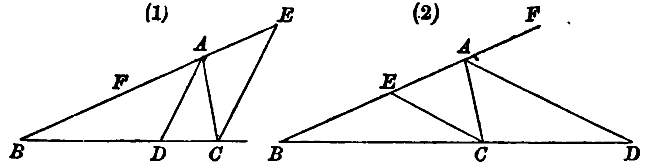
7. Take D, E, the middle points of the sides CA, CB of a triangle; join D and E, and draw AE, BD, intersecting in O; then the areas of the triangles DOE, EOB, BOA are in continued proportion.

PROPOSITION 3. PART 1.

If an angle of a triangle be bisected internally or externally by a straight line which cuts the opposite side or that side produced, the ratio of the segments of that side is equal to the ratio of the other sides of the triangle.

Let the angle BAC of the triangle ABC be bisected internally or externally by the straight line AD which cuts in D the opposite side BC (fig. 1) or BC produced (fig. 2):
it is required to prove that BD is to DC as BA to AC.

CONSTRUCTION. Through C draw CE parallel to DA to meet BA produced or BA in E; and take in BA or BA produced a point F on the side of A away from E.



PROOF. Because AC intersects the parallels AD, EC,
the angle DAC is equal to the angle ACE; (I. Prop. 29.)
and because FAE intersects the parallels AD, EC,
the angle FAD is equal to the angle AEC. (I. Prop. 29.)
And the angle DAC is equal to the angle FAD; (Hypothesis.)
therefore the angle AEC is equal to the angle ACE;
therefore AC is equal to AE. (I. Prop. 6.)

Now because AD is parallel to EC, one of the sides of the triangle BEC,
BD is to DC as BA to AE; (Prop. 2.)
and AC has been proved equal to AE;
therefore BD is to DC as BA to AC.

Wherefore, *if an angle &c.*

EXERCISES.

1. ABC is a triangle which has its base BC bisected in D. DE, DF bisect the angles ADC, ADB meeting AC, AB in E, F. Prove that EF is parallel to BC.

2. If AD bisect the angle BAC, and meet BC in D, and DE, DF bisect the angles ADC, ADB and meet AC, AB in E, F respectively, then the triangle BEF is to the triangle CEF as BA is to AC.

3. An internal point O is joined to the vertices of a triangle ABC. The bisectors of the angles BOC, COA, AOB meet BC, CA, AB respectively in D, E, F: prove that the ratio compounded of the ratios AE to EC, CD to DB, and BF to FA is unity.

4. One circle touches another internally at O. A straight line touches the inner circle at C, and meets the outer one in A, B: prove that OA is to OB as AC to CB.

5. The angle A of a triangle ABC is bisected by AD which cuts the base at D, and O is the middle point of BC: shew that OD has the same ratio to OB that the difference of the sides has to their sum.

6. AD and AE bisect the interior and the exterior angles at A of a triangle ABC, and meet the base at D and E; and O is the middle point of BC: shew that OB is a mean proportional between OD and OE.

7. If A, B, C be three points in a straight line, and D a point at which AB and BC subtend equal angles, then the locus of D is a circle.

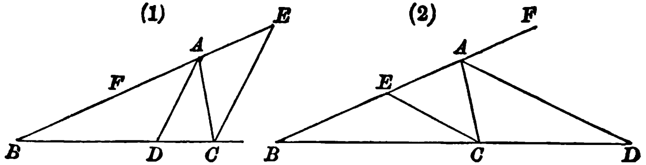
PROPOSITION 3. PART 2.

If a straight line drawn through a vertex of a triangle cut the opposite side or that side produced, so that the ratio of the segments of that side is equal to the ratio of the other sides of the triangle, the straight line bisects the vertical angle internally or externally.

Let the straight line AD drawn through A one of the vertices of the triangle ABC cut the opposite side BC or BC produced in D, so that BD is to DC as BA to AC:

it is required to prove that AD bisects the angle at A internally or externally.

CONSTRUCTION. Through C draw CE parallel to DA to meet BA produced (fig. 1) or BA (fig. 2) in E; and take in BA or BA produced a point F on the side of A away from E.



PROOF. Because DA is parallel to CE one of the sides of the triangle BEC,

BD is to DC as BA to AE. (Prop. 2.)

And BD is to DC as BA to AC; (Hypothesis.)

therefore BA is to AC as BA to AE; (V. Prop. 5.)

therefore AE is equal to AC; (V. Prop. 3.)

therefore the angle ACE is equal to the angle AEC. (I. Prop. 5.)

Again, because AC intersects the parallels AD, EC,

the angle DAC is equal to the angle ACE; (I. Prop. 29.)

and because FAE intersects the parallels AD, EC,

the angle FAD is equal to the angle AEC; (I. Prop. 29.)

therefore the angle DAC is equal to the angle FAD.

Wherefore, *if a straight line* &c.

EXERCISES.

1. The bisector of the angle BAC of a triangle ABC meets BC in D; a straight line EGF parallel to BC meets AB, AD, AC in E, G, F respectively; prove that EG is to GF as BD to DC.

2. The sides AB, AC of a given triangle ABC are produced to any points D, E, so that DE is parallel to BC. The straight line DE is divided at F so that DF is to FE as BD is to CE: shew that the locus of F is a straight line.

3. If a chord of a circle AB be divided at C so that AC is to CB as AP to PB, where P is a point on the circle: then a circle can be described to touch AB at C and the given circle at P.

4. $ABCD$ is a quadrilateral: if the bisectors of the angles at A and C meet in BD, then the bisectors of the angles at B and D meet in AC.

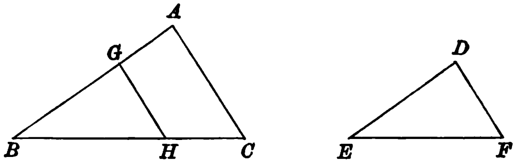
5. If A, B, C, D be four points in order on a straight line, such that AB is to BC as AD to DC, and P be any point on the circle described on BD as diameter, then PB, PD are the bisectors of the angle APC.

PROPOSITION 4.

If two triangles be equiangular to one another, they are similar.

Let the triangles ABC, DEF be equiangular to one another:
it is required to prove that the ratios
AB to DE, BC to EF and CA to FD are equal.

Construction. Of the two lines BA, ED let BA be the greater*. In BA take BG equal to ED, and in BC take BH equal to EF; and draw GH.



Proof. Because in the triangles GBH, DEF,
BG is equal to ED, and BH to EF,
and the angle GBH to the angle DEF,
the triangles are equal in all respects; (I. Prop. 4.)
therefore the angle BGH is equal to the angle EDF;
and the angle EDF is equal to the angle BAC;
(Hypothesis.)
Hence the angle BGH is equal to the angle BAC,
and AC is parallel to GH; (I. Prop. 28.)
therefore BA is to BG as BC to BH;
(Prop. 2, Part 1, Coroll.)
and GB is equal to DE, and BH to EF;
therefore AB is to DE as BC to EF.
Similarly it can be proved that either of these ratios is equal to the ratio CA to FD.

Wherefore, *if two triangles* &c.

* The case when BA is equal to ED has already been dealt with. (I. Prop. 26.)

EXERCISES.

1. A common tangent to two circles cuts the straight line joining the centres externally or internally in the ratio of the radii.

2. If *AB*, *CD*, two parallel straight lines, be divided proportionally by *P*, *Q*, so that *AP* is to *PB* as *CQ* to *QD*, then *AC*, *PQ*, *BD* meet in a point.

3. *ABCD* is a parallelogram; *P* and *Q* are points in a straight line parallel to *AB*; *PA* and *QB* meet at *R*, and *PD* and *QC* meet at *S*; shew that *RS* is parallel to *AD*.

4. The tangents at the points *P* and *Q* of a circle intersect in *T*: if from any other point *R* of the circle the perpendiculars *RM*, *RN* be drawn to the tangents *TP* and *TQ*, and the perpendicular *RL* be drawn to the chord *PQ*, then *RL* is a mean proportional between *RM* and *RN*.

5. A straight line, parallel to the side *BC* of a triangle *ABC*, meets the sides *AB*, *AC* (or those sides produced) at *D* and *E*. On *DE* is constructed a parallelogram *DEFG*, and the straight lines *BG*, *CF* (produced if necessary) meet each other at *S*. Prove that *AS* is parallel to *DG* or *EF*.

6. Inscribe an equilateral triangle in a given triangle, so as to have one side parallel to a side of the given triangle.

7. If two triangles have their bases equal and in the same straight line, and also have their vertices on a parallel straight line, any straight line parallel to their bases will cut off equal areas from the two triangles.

8. In a given triangle *ABC* draw a straight line *PQ* parallel to *AB* meeting *AC*, *BC* in *P*, *Q*, so that *PQ* may be a mean proportional between *BQ*, *QC*.

9. Two circles intersect at *A*, and a straight line is drawn bisecting the angle between the tangents at *A*. Prove that the segments of the line cut off by the circles are proportional to the radii.

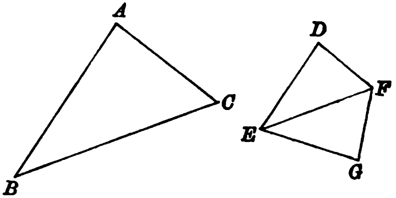
10. If *ACB*, *BCD* be equal angles, and *DB* be perpendicular to *BC* and *BA* to *AC*, then the triangle *DBC* is to the triangle *ABC* as *DC* is to *CA*.

PROPOSITION 5.

If the ratios of the three sides of one triangle to the three sides of another triangle be equal, the triangles are equiangular to one another.

Let the given triangles ABC, DEF be such that the ratios AB to DE, BC to EF and CA to FD are equal:
it is required to prove that the triangles ABC, DEF are equiangular to one another.

CONSTRUCTION. On the side of EF away from D draw EG, FG making the angles FEG, EFG equal to the angles CBA, BCA respectively. (I. Prop. 23.)



PROOF. Because in the triangles ABC, GEF,
the angles ABC, BCA are equal to the angles GEF, EFG,
the triangles ABC, GEF are equiangular to one another,
(I. Prop. 32.)
and therefore AB is to GE as BC to EF. (Prop. 4.)
And AB is to DE as BC to EF; (Hypothesis.)
therefore AB is to GE as AB to DE; (V. Prop. 5.)
therefore GE is equal to DE. (V. Prop. 3.)
Similarly it can be proved that GF is equal to DF.
Then because in the triangles DEF, GEF,
DE, EF, FD are equal to GE, EF, FG respectively,
the triangles are equal in all respects; (I. Prop. 8.)
therefore the triangles DEF, GEF are equiangular to one another;
and the triangle GEF was constructed so as to be equiangular to the triangle ABC;
therefore the triangles ABC, DEF are equiangular to one another.

Wherefore, *if the ratios* &c.

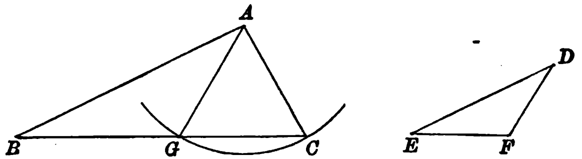
PROPOSITION 5. A.

If one pair of angles of two triangles be equal and another pair of angles be supplementary, the ratios of the sides opposite to these pairs of angles are equal.

Let ABC, DEF be two triangles in which the angles ABC, DEF are equal, and the angles ACB, DFE are supplementary:

it is required to prove that AB is to DE as AC to DF.

CONSTRUCTION. Of the two angles ACB, DFE, let ACB be the less. With A as centre and AC as radius describe a circle cutting BC in G; and draw AG.



PROOF. Because AC is equal to AG,

the angle AGC is equal to the angle ACG; (I. Prop. 5.)

and the angle AGB is the supplement of the angle AGC,

and the angle DFE is the supplement of the angle ACB; (Hypothesis.)

therefore the angle AGB is equal to the angle DFE;

and the angle ABG is equal to the angle DEF; (Hypothesis.)

therefore the triangles ABG, DEF are equiangular to one another; (I. Prop. 32.)

therefore AB is to DE as AG to DF; (Prop. 4.)

and AC is equal to AG; (Constr.)

therefore AB is to DE as AC to DF.

Wherefore, *if one pair of angles*, &c.

PROPOSITION 6.

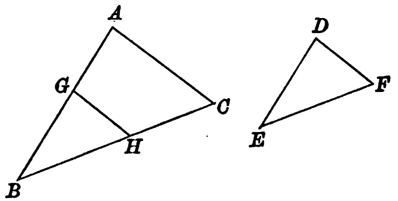
If the ratios of two sides of one triangle to two sides of another triangle be equal, and also the angles contained by those sides be equal, the triangles are equiangular to one another.

Let *ABC*, *DEF* be two triangles in which *AB* is to *DE* as *BC* to *EF*,

and the angle *ABC* is equal to the angle *DEF*:

it is required to prove that the triangles *ABC*, *DEF* are equiangular to one another.

Construction. Of the two lines *BA*, *ED* let *BA* be the greater*. In *BA* take *BG* equal to *ED*, and in *BC* take *BH* equal to *EF*; and draw *GH*.



Proof. Because *BA* is to *ED* as *BC* to *EF*,
and *BG* is equal to *ED*, and *BH* to *EF*,
therefore *BA* is to *BG* as *BC* to *BH*;
therefore *GH* is parallel to *AC*, (Prop. 2, Part 2, Coroll.)
and the angle *BGH* is equal to the angle *BAC*.
(I. Prop. 29.)

Again, because in the triangles *DEF*, *GBH*,
ED is equal to *BG* and *EF* to *BH*,
and the angle *DEF* to the angle *GBH*,
the triangles are equal in all respects; (I. Prop. 4.)

* The case when *BA* is equal to *ED* has already been dealt with. (I. Prop. 4.)

therefore the angle EDF is equal to the angle BGH,
and therefore to the angle BAC.

And the angles at B and E are equal; (Hypothesis.)
therefore the triangles ABC, DEF are equiangular to one another. (I. Prop. 32.)

Wherefore, *if the ratios* &c.

EXERCISES.

1. Shew that the locus of the middle points of straight lines parallel to the base of a triangle and terminated by its sides is a straight line.

2. CAB, CEB are two triangles having the angle B common and the sides CA, CE equal; if BAE be produced to D and ED be taken a third proportional to BA, AC, then the triangle BDC is similar to the triangle BAC.

3. From a point E in the common base of the triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD in F and G; shew that FG, CD are parallel.

4. C is a point in a given straight line AB, and AB is produced to O, so that CO is a mean proportional between AO and BO. If P be any point on a circle described with centre O and radius OC, then the angles APC, BPC are equal.

5. If a point O be taken within a parallelogram $ABCD$, such that the angles OBA, ODA are equal, then the angles OAD, OCD are equal.

6. If two points P, Q be such that when four perpendiculars PM, Pm, QN, Qn are dropped upon the straight lines AMN, Amn, PM is to Pm as QN to Qn, then P and Q lie on a straight line through A.

7. If on the three sides of any triangle, equilateral triangles be described either all externally or all internally, the centres of the circles inscribed in these triangles are the vertices of an equilateral triangle.

8. The straight line OP joining a fixed point O to a variable point P on a fixed circle is divided in Q in a constant ratio; prove that the locus of Q is a circle.

9. Given the base and the vertical angle of a triangle, find the locus of the intersection of bisectors of sides.

PROPOSITION 7.

If the ratios of two sides of one triangle to two sides of another triangle be equal, and also the angles opposite to one pair of these sides be equal, the angles opposite to the other pair of sides are equal or supplementary.

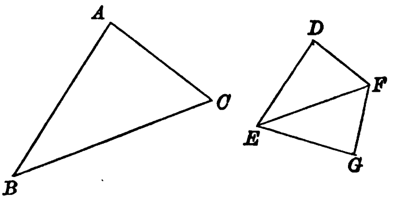
Let ABC, DEF be two triangles, in which

AB is to DE as BC to EF,

and the angle BAC is equal to the angle EDF:

it is required to prove that the angles ACB, DFE are either equal or supplementary.

CONSTRUCTION. On the side of EF away from D, draw EG making the angle FEG equal to the angle CBA, and draw FG making the angle EFG equal to the angle BCA. (I. Prop. 23.)



PROOF. Because the triangles ABC, GEF are equiangular to one another, (I. Prop. 32.)

AB is to GE as AC to GF; (Prop. 4.)

and AB is to DE as BC to EF; (Hypothesis.)

therefore AB is to GE as AB to DE, (V. Prop. 5.)

and GE is equal to DE. (V. Prop. 3.)

Now because in the triangles GEF, DEF,

GE is equal to DE and EF to EF,

and the angle EGF to the angle EDF;

(for each is equal to the angle at A)

therefore the angles GFE, DFE are either equal or supplementary; (I. Prop. 26, A.)

and the angle GFE is equal to the angle ACB; (Constr.)

therefore the angles ACB, DFE are either equal or supplementary.

Wherefore, *if the ratios* &c.

COROLLARY. *When two of the ratios of a side of one triangle to the corresponding side of another triangle are equal, and also the angles opposite to one pair of these sides equal, the triangles are equiangular to one another, provided that of the angles opposite to the second pair of sides,*

(1) *each be less than a right angle,*

(2) *each be greater than a right angle,*

or (3) *one of them be a right angle.*

(I. Prop. 26 A, Coroll.)

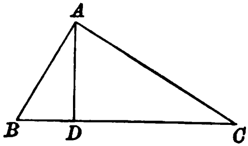
EXERCISE.

Prove that, if $ABCD$, $EFGH$ be two quadrilaterals, such that the angles ABC, ADC are equal to the angles EFG, EHG respectively, and the ratios AB to EF, BC to FG, CD to GH are equal, and if the angles BAD, FEH be both acute angles, then the quadrilaterals are similar.

PROPOSITION 8.

In a right-angled triangle, if a perpendicular be drawn from the opposite vertex to the hypotenuse, the perpendicular is a mean proportional between the segments of the hypotenuse, and each of the sides of the triangle is a mean proportional between the hypotenuse and the segment of it adjacent to that side.

Let ABC be a right-angled triangle, and let AD be drawn perpendicular to the hypotenuse BC:
it is required to prove that BD is to DA as AD to DC, that BC is to BA as BA to BD, and that BC is to CA as CA to CD.



PROOF. Because in the triangles ABC, DBA,
the right angle BAC is equal to the right angle BDA,
and the angle ABC is equal to the angle DBA,
therefore the triangles ABC, DBA are equiangular to one another. (I. Prop. 32.)
Similarly it can be proved that the triangles DAC, ABC are equiangular to one another.
Therefore the triangles DBA, DAC are equiangular to one another.

Now, because the triangles DBA, DAC are equiangular to one another,
BD is to DA as AD to DC; (Prop. 4.)
and because the triangles ABC, DBA are equiangular to one another,
BC is to BA as BA to BD; (Prop. 4.)
and because the triangles ABC, DAC are equiangular to one another,
BC is to CA as AC to CD. (Prop. 4.)

Wherefore, *in a right-angled triangle* &c.

EXERCISES.

1. If the perpendicular drawn from the vertex of a triangle to the base be a mean proportional between the segments of the base, the triangle is right-angled.

2. If a triangle whose sides are unequal can be divided into two similar triangles by a straight line joining the vertex to a point in the base, the vertical angle must be a right angle.

3. If CD, CE, the internal and the external bisectors of the angle at C in a triangle ABC having a right angle at A, cut BA in D and E respectively, then AC is a mean proportional between AD, AE.

4. A perpendicular AD is drawn to the hypotenuse BC of a right-angled triangle from the opposite vertex A: and perpendiculars DE, DF are drawn from D to the sides AB, AC; prove that a circle will pass through the four points B, E, F, C.

5. On the tangent to a circle at A two points C and B are taken such that AC is equal to CB: the straight lines joining B, C to F, the opposite extremity of the diameter through A, cut the circle in D, E respectively; prove that AE is to ED as FA to FD.

6. A chord CD is drawn parallel to a diameter AB of a circle, and AC, AD are produced to cut the tangent at B in E, F respectively; prove that the sum of the rectangles AC, CE and AD, DF is equal to the square on AB.

7. If A be a point outside a circle and B be the middle point of the chord of contact of tangents drawn from A, and P, Q be any two points on the circle, then PA is to QA as PB to QB.

8. Two circles intersect in A, B; from B perpendiculars BE, BF are drawn to their diameters AC, AD; prove that C, E, F, D lie on a circle, which is cut at right angles by the circle whose centre is A and radius AB.

9. The circumference of one circle passes through the centre of another circle. If from any point of the former circle two straight lines be drawn to touch the latter circle, the straight line joining the points of contact is bisected by the common chord of the two circles.

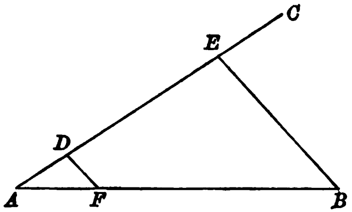
PROPOSITION 9.

From a given finite straight line to cut off any aliquot part required.

Let AB be the given finite straight line:
it is required to cut off from AB a given aliquot part, say the n^{th} part.

CONSTRUCTION. From A draw any straight line AC making an angle with AB, and in it take any point D, and cut off AE the same multiple of AD that AB is of the part to be cut off, i.e. take AE equal to n times AD.

Draw EB, and draw DF parallel to it meeting AB in F:
then AF is the part required.



PROOF. Because FD is parallel to BE, one of the sides of the triangle ABE,

AB is to AF as AE to AD; (Prop. 2, Part 1, Coroll.)
and AE is equal to n times AD;
therefore AB is equal to n times AF.
Therefore AF is the n^{th} part of AB.

Wherefore, *from the given straight line AB, AF the part required has been cut off.*

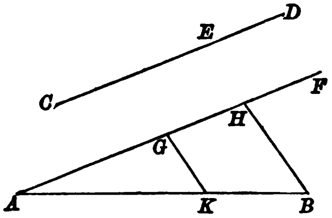
PROPOSITION 10.

To divide a given finite straight line similarly to a given divided straight line.

Let AB be a given straight line and CD another given straight line divided in E:
it is required to divide AB similarly to CD.

CONSTRUCTION. Draw AF making an angle with AB; cut off AG, GH equal to CE, ED respectively.

Draw HB, and draw GK parallel to HB meeting AB in K:
then AB is divided at K similarly to CD at E.



PROOF. Because GK is parallel to HB one of the sides of the triangle AHB,

AK is to KB as AG to GH; (Prop. 2.)

and AG is equal to CE, and GH to ED. (Constr.)

Therefore AK is to KB as CE to ED.

Wherefore, *the straight line AB has been divided at K similarly to the straight line CD at E.*

EXERCISES.

1. If three straight lines passing through a point O cut two parallel straight lines ABC, PQR in A, P; B, Q; C, R, then the lines AC, PR are similarly divided in B, Q.

2. Draw a straight line through a given point A, so that the perpendiculars upon it from two other given points B and C may be in a given ratio.

3. Draw through two given points on a circle two parallel chords which shall have a given ratio to one another.

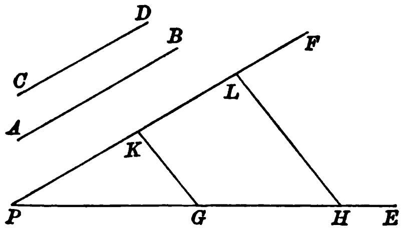
PROPOSITION 11.

To find a third proportional to two given finite straight lines.

Let AB, CD be two given straight lines:
it is required to find a third proportional to AB, CD.

Construction. Draw from any point P a pair of straight lines PE, PF making an angle with one another, and from PE cut off PG, GH equal to AB, CD respectively and from PF cut off PK equal to CD.

Draw GK and draw HL parallel to GK meeting PF in L:
then KL is a third proportional to AB, CD.



Proof. Because GK is parallel to HL one of the sides of the triangle PHL,

PG is to GH as PK to KL; (Prop. 2.)

and PG is equal to AB and GH and PK are each equal to CD;

therefore AB is to CD as CD to KL.

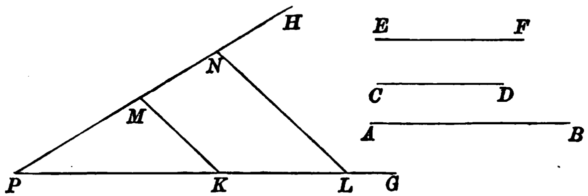
Wherefore *to the two given straight lines AB, CD a third proportional KL has been found.*

PROPOSITION 12.

To find a fourth proportional to three given straight lines.

Let *AB*, *CD*, *EF* be three given straight lines:
it is required to find a fourth proportional to *AB*, *CD*, *EF*.

CONSTRUCTION. Draw from any point *P* a pair of straight lines *PG*, *PH* making an angle with one another; and from *PG* cut off *PK*, *KL* equal to *AB*, *CD* respectively,
and from *PH* cut off *PM* equal to *EF*.
Draw *KM*, and draw *LN* parallel to *KM* meeting *PH* in *N*:
then *MN* is a fourth proportional to *AB*, *CD*, *EF*.



PROOF. Because *KM* is parallel to *LN* one of the sides of the triangle *PLN*,
PK is to *KL* as *PM* to *MN*; (Prop. 2.)
and *PK* is equal to *AB*, *KL* to *CD*, and *PM* to *EF*;
therefore *AB* is to *CD* as *EF* to *MN*.

Wherefore *to the three given straight lines AB, CD, EF, a fourth proportional MN has been found.*

EXERCISE.

1. *C* is a point on a straight line *AB*; find a point *D* in *AB* produced, such that *DA* is to *DB* as *CA* to *CB*.

PROPOSITION 13.

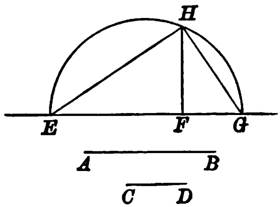
To find a mean proportional between two given straight lines.

Let AB, CD be two given straight lines:
it is required to find a mean proportional between AB and CD.

CONSTRUCTION. Draw any straight line and from it cut off EF, FG equal to AB, CD respectively.

Describe a circle on EG as diameter and draw FH at right angles to EG meeting the circle in H:
then FH is a mean proportional between AB and CD.

Draw EH, HG.



PROOF. Because EHG is a semicircle,
the angle EHG is a right angle; (III. Prop. 31.)
and because HF is the perpendicular from H on the hypotenuse of the right-angled triangle EHG,
EF is to FH as FH to FG; (Prop. 8.)
and EF is equal to AB and FG to CD;
therefore AB is to FH as FH to CD.

Wherefore, *between the two given straight lines AB, CD a mean proportional FH has been found.*

EXERCISES.

1. Find a mean proportional between two given straight lines by the use of the theorem of Proposition 37 of Book III.

2. Divide a given finite straight line into two parts, so that their mean proportional may be of given length.

3. Construct an isosceles triangle equal to a given triangle and having the vertical angle equal to one of the angles of the given triangle.

4. Find a third proportional to two given straight lines by a method similar to that of Proposition 13.

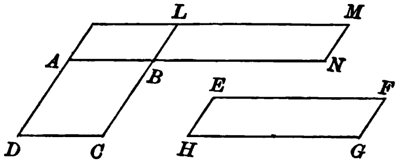
DEFINITION. *If the ratio of a side of one polygon to a side of another polygon be equal to the ratio of an adjacent side of the second to an adjacent side of the first, those sides are said to be* **reciprocally proportional.**

PROPOSITION 14. PART 1.

If two parallelograms, which have a pair of equal angles, be equal in area, their sides about the equal angles are reciprocally proportional.

Let *ABCD*, *EFGH* be two parallelograms, which have the angles at *B* and *H* equal, and which are equal in area: it is required to prove that *AB* is to *HG* as *EH* to *BC*.

CONSTRUCTION. From *AB*, *CB* produced cut off *BN*, *BL* equal to *HG*, *HE*, and complete the parallelograms *AL*, *LN*.



PROOF. Because in the parallelograms *LN*, *EG*,
LB is equal to *EH*, and *BN* to *HG*,
and the angle *LBN* to the angle *EHG*,
therefore the parallelograms *LN*, *EG* are equal in area.
(I. Props. 4 and 34.)

And the area of *EG* is equal to the area of *AC*;
therefore the area of *AL* is to the area of *LN* as the area of *AL* to the area of *AC*.

And *AB* is to *BN* as the area of *AL* to the area of *LN*,
and *LB* is to *BC* as the area of *AL* to the area of *AC*;
(Prop. 1, Coroll. 3.)

therefore *AB* is to *BN* as *LB* to *BC*, (V. Prop. 5.)
that is, *AB* is to *HG* as *EH* to *BC*.

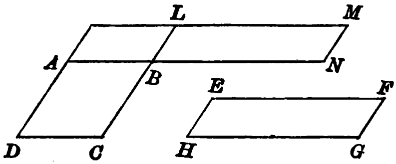
Wherefore, *if two parallelograms* &c.

PROPOSITION 14. PART 2.

If two parallelograms, which have a pair of equal angles, have their sides about the equal angles reciprocally proportional, the parallelograms are equal in area.

Let *ABCD*, *EFGH* be two parallelograms, which have the angles at *B*, *H* equal and in which *AB* is to *HG* as *EH* to *BC*:
it is required to prove that the parallelograms *ABCD*, *EFGH* are equal in area.

CONSTRUCTION. From *AB*, *CB* produced cut off *BN*, *BL* equal to *HG*, *HE* and complete the parallelograms *AL*, *LN*.



PROOF. Because in the parallelograms *LN*, *EG*,
LB is equal to *EH*, and *BN* to *HG*,
and the angle *LBN* to the angle *EHG*,
therefore the parallelograms *LN*, *EG* are equal in area.
(I. Props. 4 and 34.)
Because *AB* is to *HG* as *EH* to *BC*,
and *BN* is equal to *HG* and *LB* to *EH*,
therefore *AB* is to *BN* as *LB* to *BC*.
And *AB* is to *BN* as the area of *AL* to the area of *LN*,
and *LB* is to *BC* as the area of *AL* to the area of *AC*;
(Prop. 1, Coroll. 3.)
therefore the area of *AL* is to the area of *LN* as the area of *AL* to the area of *AC*. (V. Prop. 5.)
Therefore the area of *LN* is equal to the area of *AC*.
And the parallelograms *LN*, *EG* are equal in area;
therefore the parallelograms *AC*, *EG* are equal in area.

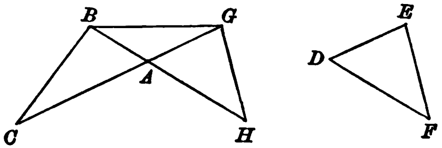
Wherefore, *if two parallelograms* &c.

PROPOSITION 15. PART 1.

If two triangles, which have a pair of equal angles, be equal in area, their sides about the equal angles are reciprocally proportional.

Let *ABC*, *DEF* be two triangles, which have the angles at *A*, *D* equal and which are equal in area:
it is required to prove that *BA* is to *DF* as *ED* to *AC*.

CONSTRUCTION. From *BA*, *CA* produced cut off *AH*, *AG* equal to *DF*, *DE* respectively, and draw *BG*, *GH*.



PROOF. Because in the triangles *AGH*, *DEF*,
GA is equal to *ED*, and *AH* to *DF*,
and the angle *GAH* to the angle *EDF*,
the triangles are equal in all respects. (I. Prop. 4.)
And the area of *DEF* is equal to the area of *ABC*;
therefore the area of *AGB* is to the area of *AGH* as the area of *ABG* tó the area of *ABC*.
And *BA* is to *AH* as the area of *ABG* to the area of *GAH*,
and *GA* is to *AC* as the area of *ABG* to the area of *ABC*; (Prop. 1.)
therefore *BA* is to *AH* as *GA* to *AC*; (V. Prop. 5.)
and *AH* is equal to *DF* and *GA* to *ED*; (Constr.)
therefore *BA* is to *DF* as *ED* to *AC*.

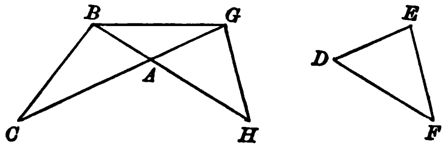
Wherefore, *if two triangles* &c.

PROPOSITION 15. PART 2.

If two triangles, which have a pair of equal angles, have their sides about the equal angles reciprocally proportional, the triangles are equal in area.

Let *ABC*, *DEF* be two triangles, which have the angles at *A*, *D* equal and in which *BA* is to *DF* as *ED* to *AC*:
it is required to prove that the triangles *ABC*, *DEF* are equal in area.

CONSTRUCTION. From *BA*, *CA* produced cut off *AH*, *AG* equal to *DF*, *DE* respectively, and draw *BG*, *GH*.



PROOF. Because in the triangles *AGH*, *DEF*,
GA is equal to *ED*, and *AH* to *DF*,
and the angle *GAH* to the angle *EDF*,
the triangles are equal in all respects. (I. Prop. 4.)
Because *BA* is to *DF* as *ED* to *AC*,
and *AH* is equal to *DF* and *GA* to *ED*;
therefore *BA* is to *AH* as *GA* to *AC*.
And *BA* is to *AH* as the area of *GBA* to the area of *AGH*, and *GA* is to *AC* as the area of *GBA* to the area of *ABC*; (Prop. 1.)
therefore the area of *GBA* is to the area of *AGH* as the area of *GBA* to the area of *ABC*; (V. Prop. 5.)
therefore the area of *AGH* is equal to the area of *ABC*;
and the triangles *AGH*, *DEF*, being equal in all respects, are equal in area;
therefore the triangles *ABC*, *DEF* are equal in area.

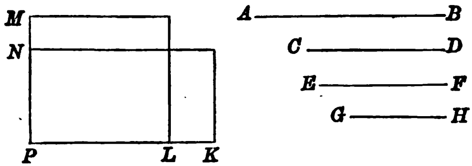
Wherefore, *if two triangles &c.*

PROPOSITION 16. PART 1.

If four straight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means.

Let the straight lines *AB*, *CD*, *EF*, *GH* be proportionals, so that *AB* is to *CD* as *EF* to *GH*:
it is required to prove that the rectangle contained by *AB* and *GH* is equal to the rectangle contained by *CD* and *EF*.

CONSTRUCTION. From any point *P* draw two straight lines at right angles and on one cut off *PK*, *PL* equal to *AB*, *CD* respectively; and on the other cut off *PM*, *PN* equal to *EF*, *GH* respectively; and complete the rectangles *ML*, *NK*.



PROOF. Because *AB* is to *CD* as *EF* to *GH*,
and *PK*, *PL*, *PM*, *PN*, are equal to *AB*, *CD*, *EF*, *GH* respectively,
therefore *PK* is to *PL* as *PM* to *PN*.
And the angle at *P* is common to the two rectangles *NK*, *ML*;
therefore the rectangle *NK* is equal to the rectangle *ML*; (Prop. 14, Part 2.)
and *NK* is contained by *PK*, *PN* and *ML* is contained by *PL*, *PM*.
Therefore the rectangle contained by *AB*, *GH* is equal to the rectangle contained by *CD*, *EF*.

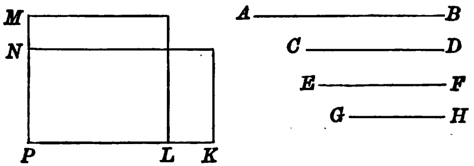
Wherefore, *if four straight lines* &c.

PROPOSITION 16. PART 2.

If the rectangle contained by the first and the fourth of four given straight lines be equal to the rectangle contained by the second and the third, the four lines are proportionals.

Let AB, CD, EF, GH be four straight lines, such that the rectangle contained by AB and GH is equal to the rectangle contained by CD and EF:
it is required to prove that AB is to CD as EF to GH.

CONSTRUCTION. From any point P draw two straight lines at right angles, and on one cut off PK, PL equal to AB, CD respectively; and on the other cut off PM, PN equal to EF, GH respectively; and complete the rectangles ML, NK.



PROOF. Because the rectangle contained by AB and GH is equal to the rectangle contained by CD and EF,
and PK, PL, PM, PN are equal to AB, CD, EF, GH respectively,
the rectangle contained by PK and PN is equal to the rectangle contained by PL and PM,
that is, the rectangle NK is equal to the rectangle ML;
therefore PK is to PL as PM to PN; (Prop. 14, Part 1.)
and PK is equal to AB, PL to CD, PM to EF and PN to GH; (Constr.)
therefore AB is to CD as EF to GH.

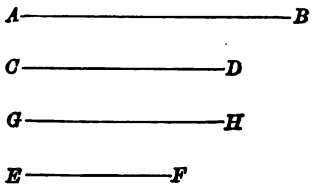
Wherefore, *if the rectangle* &c.

PROPOSITION 17. PART 1.

If three straight lines be proportionals, the rectangle contained by the extremes is equal to the square on the mean.

Let the three straight lines *AB*, *CD*, *EF* be proportionals, so that *AB* is to *CD* as *CD* to *EF*:
it is required to prove that the rectangle contained by *AB* and *EF* is equal to the square on *CD*.

CONSTRUCTION. Draw a straight line *GH* equal to *CD*.



PROOF. Because *AB* is to *CD* as *CD* to *EF*, (Hypothesis.)
and *GH* is equal to *CD*; (Constr.)
therefore *AB* is to *CD* as *GH* to *EF*;
therefore the rectangle contained by *AB* and *EF* is equal to the rectangle contained by *CD* and *GH*,
(Prop. 16, Part 1.)
which is equal to the square on *CD*, for *GH* is equal to *CD*.

Wherefore, *if three straight lines* &c.

EXERCISE.

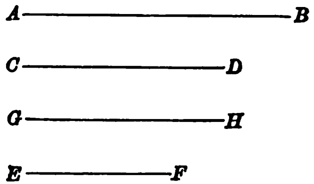
1. A square is inscribed in a right-angled triangle *ABC*, so that two corners *D*, *E* lie on the hypotenuse *AB* and the other two on the sides *BC*, *CA*; prove that the square is equal to the rectangle *AD*, *EB*.

PROPOSITION 17. PART 2.

If the rectangle contained by the first and the third of three given straight lines be equal to the square on the second, the three straight lines are proportionals.

Let AB, CD, EF be three given straight lines, such that the rectangle contained by AB and EF is equal to the square on CD:
it is required to prove that AB is to CD as CD to EF.

CONSTRUCTION. Draw a straight line GH equal to CD.



PROOF. Because GH is equal to CD,
the square on CD is equal to the rectangle contained by CD and GH;
therefore the rectangle contained by AB and EF, which is equal to the square on CD, is equal to the rectangle contained by CD and GH;
therefore AB is to CD as GH to EF; (Prop. 16, Part 2.)
and GH is equal to CD;
therefore AB is to CD as CD to EF.

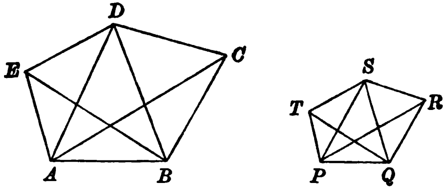
Wherefore, *if the rectangle* &c.

PROPOSITION 18.

On a given straight line to construct a polygon similar to a given polygon so that the given straight line may correspond to a given side of the given polygon.

Let AB be the given straight line, and $PQRST$ the given polygon:
it is required to construct on AB a polygon similar to $PQRST$ so that AB, PQ may be corresponding sides.

Construction. From P, Q the extremities of PQ, draw the diagonals PR, PS, QS, QT of the polygon $PQRST$. At A, B on the same side of AB make the angles BAC, BAD, BAE equal to the angles QPR, QPS, QPT respectively, and the angles ABC, ABD, ABE to the angles PQR, PQS, PQT respectively, and draw CD, DE:
then $ABCDE$ is a polygon constructed as required.



Proof. Because the triangles ABC, PQR are equiangular to one another, (Constr.)
AC is to PR as AB to PQ; (Prop. 4.)
and because the triangles ABD, PQS are equiangular to one another, (Constr.)
AD is to PS as AB to PQ; (Prop. 4.)
therefore AD is to PS as AC to PR. (V. Prop. 5.)

Again, because in the triangles DAC, SPR,
the ratios of AD to PS and AC to PR are equal,
and the angles DAC, SPR are equal; (Constr.)
therefore the triangles DAC, SPR are equiangular to one another, and CD is to RS as AC to PR; (Prop. 6.)
therefore CD is to RS as AB to PQ. (V. Prop. 5.)

Again because the triangles ABC, PQR are equiangular to one another, (Constr.)

the angle ACB is equal to the angle PRQ;

and because the triangles DAC, SPR have been proved equiangular to one another,

the angle ACD is equal to the angle PRS;

therefore the angle BCD is equal to the angle QRS.

Similarly it can be proved that the ratio of any other corresponding pair of sides of the polygons $ABCDE$, $PQRST$ is equal to that of AB to PQ, and that any corresponding pair of angles are equal.

Wherefore, *on the given straight line* AB, *the polygon* $ABCDE$ *has been constructed similar to the given polygon* $PQRST$, *so that* AB, PQ *are corresponding sides.*

EXERCISES.

1. Given the length of the line joining the middle point of a side of a square with an end of the opposite side; determine, by any method, the length of a diagonal of the square.

2. Inscribe in a given triangle a second triangle so that its sides may be parallel to three given straight lines.
In how many ways can this be done?

3. In a triangle ABC inscribe a square so that two of its vertices may be on BC and the other two on AB, AC.

4. In a semicircle inscribe a square, so that two corners may lie in the diameter and two on the circumference.

5. In a given sector of a circle inscribe a square so that two corners may lie on the arc and one on each of the bounding radii.

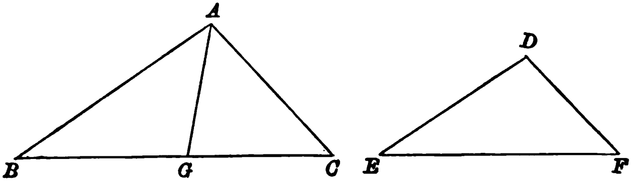
6. In a given sector inscribe a square so that two corners may be on one of the bounding radii, one on the other bounding radius and one on the arc.

PROPOSITION 19.

Similar triangles are to one another in the ratio duplicate of the ratio of two corresponding sides.

Let *ABC*, *DEF* be similar triangles and *BC*, *EF* be corresponding sides:
it is required to prove that the triangle *ABC* is to the triangle *DEF*, in the ratio duplicate of the ratio of *BC* to *EF*.

CONSTRUCTION. Find a third proportional to *BC*, *EF* and from *BC* cut off *BG* equal to it. Draw *AG*.



PROOF. Because the triangles *ABC*, *DEF* are similar,
AB is to *DE* as *BC* to *EF*; (Prop. 4.)
and *BC* is to *EF* as *EF* to *BG*; (Constr.)
therefore *AB* is to *DE* as *EF* to *BG*; (V. Prop. 5.)
and the angle *ABG* is equal to the angle *DEF*;
therefore the triangles *ABG*, *DEF* are equal in area.
(Prop. 15, Part 2.)

And the triangle *ABC* is to the triangle *ABG* as *BC* to *BG*;
therefore the triangle *ABC* is to the triangle *DEF* as *BC* to *BG*;

And because *BC* is to *EF* as *EF* to *BG*,
BC has to *BG* the ratio duplicate of the ratio of *BC* to *EF*.
(V. Def. 9.)

Therefore the triangle *ABC* has to the triangle *DEF* the ratio duplicate of the ratio of *BC* to *EF*.

Wherefore, *similar triangles* &c.

COROLLARY. *If three straight lines be proportionals, the first is to the third as any triangle on the first to a similar triangle on the second.*

EXERCISES.

1. Through a point within a triangle three straight lines are drawn parallel to the sides, dividing the triangle into three triangles and three parallelograms: if the three triangles be equal to each other in area, each is one-ninth of the original triangle.

2. An isosceles triangle is described having each of the angles at the base double of the third angle: if the angles at the base be bisected, and the points where the lines bisecting them meet the opposite sides be joined, the triangle will be divided into two parts having the same ratio as the base to the side of the triangle.

3. ABC is a triangle, the angle A being greater than the angle B: a point D is taken in BC, such that the angle CAD is equal to B. Prove that CD is to CB in the ratio duplicate of the ratio of AD to AB.

4. The sides BC, CA, AB of an equilateral triangle ABC are divided in the points D, E, F so that the ratios BD to DC, CE to EA and AF to FB are each equal to 2 to 1. Find the ratio of the triangle DEF to the triangle ABC.

5. If a straight line AB be produced to a point C so that AB is a mean proportional between AC and CB, then the square on AB is to the square on BC as AB to the excess of AB over BC.

6. Find a mean proportional between the areas of two similar right-angled triangles which have one of the sides containing the right angle common.

7. Bisect a given triangle by a line parallel to its base.

8. Bisect a given triangle by a line drawn perpendicular to its base.

9. Divide a given triangle into two parts, having a given ratio to one another, by a straight line parallel to one of its sides.

10. ABC is a triangle; AB is produced to E: AD is a straight line meeting BC in D: BF is parallel to ED and meets AD in F: construct a triangle similar to ABC and equal to AEF.

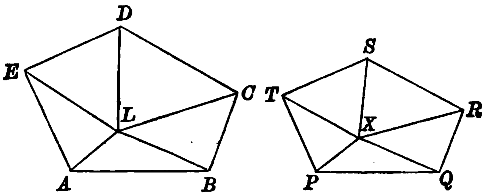
PROPOSITION 20.

A pair of similar polygons may be divided into pairs of similar triangles, each pair having the same ratio as the polygons.

Let $ABCDE$, $PQRST$ be a pair of similar polygons: it is required to prove that the polygons can be divided into pairs of similar triangles.

CONSTRUCTION. Take any point L within the polygon $ABCDE$, and draw LA, LB, LC, LD, LE.

Within the polygon $PQRST$, draw PX, QX making the angles QPX, PQX equal to the angles BAL, ABL respectively, and draw XR, XS, XT.



PROOF. Because the triangles LAB, XPQ are equiangular to one another, (Constr.)

LB is to XQ as AB to PQ; (Prop. 4.)

and because the polygons $ABCDE$, $PQRST$ are similar,

AB is to PQ as BC to QR; (Def. 2.)

therefore LB is to XQ as BC to QR. (V. Prop. 5.)

Again because the polygons $ABCDE$, $PQRST$ are similar,

the angle ABC is equal to the angle PQR;

and the angle ABL is equal to the angle PQX; (Constr.)

therefore the angle LBC is equal to the angle XQR.

Therefore the triangles LBC, XQR are equiangular to one another, (Prop. 6.)

and therefore similar. (Prop. 4.)

Similarly it can be proved that the triangles LCD, LDE, LEA are similar to the triangles XRS, XST, XTP respectively.

Again, because AB is to PQ as BC to QR;
and because the triangle LAB is to the triangle XPQ in the ratio duplicate of the ratio of AB to PQ,
and the triangle LBC is to the triangle XQR in the ratio duplicate of the ratio of BC to QR, (Prop. 19.)
therefore the triangle LAB is to the triangle XPQ as the triangle LBC to the triangle XQR. (V. Prop. 14. Coroll.)

Similarly it can be proved that each of the ratios of the triangles LCD, LDE, LEA to the triangles XRS, XST, XTP respectively is equal to the ratio of the triangle LAB to the triangle XPQ.
Therefore the polygon $ABCDE$ is to the polygon $PQRST$ as the triangle LAB to the triangle XPQ. (V. Prop. 6.)

Wherefore, *a pair of similar polygons* &c.

COROLLARY. *Similar polygons are to one another in the ratio duplicate of the ratio of two corresponding sides.*

EXERCISES.

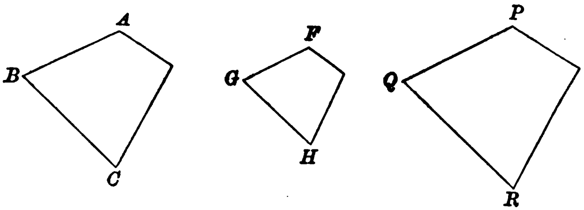
1. If ABC be a right-angled triangle and CD be drawn perpendicular to the hypotenuse, then AD is to DB as the square on AC to the square on CB.

2. If a straight line be drawn from each corner of a square to the nearer point of trisection of the next side of the square in order, so as to form a square, this square will be two-fifths of the original square. What will be the area of the new square, if the lines be drawn to the further point of trisection?

PROPOSITION 21.

Polygons which are similar to the same polygon are similar to one another.

Let each of the polygons $ABC...$, $FGH...$, be similar to the polygon $PQR...$:
it is required to prove that $ABC...$, $FGH...$ are similar to one another.



PROOF. Because the polygons $ABC...$, $PQR...$ are similar,

the angle ABC is equal to the angle PQR,
and AB is to PQ as BC to QR; (Def. 2.)

and because the polygons $FGH...$, $PQR...$ are similar,

the angle FGH is equal to the angle PQR,
and PQ is to FG as QR to GH.

Therefore the angle ABC is equal to the angle FGH,
and AB is to FG as BC to GH. (V. Prop. 14.)

Similarly it can be proved that every pair of corresponding angles of the polygons $ABC...$, $FGH...$ are equal and that the ratios of all pairs of corresponding sides are equal.

Therefore the polygons $ABC...$, $FGH...$ are similar.

Wherefore, *polygons which are similar* &c.

EXERCISES.

1. Prove that, if *ABCD*, *EFGH* be two quadrilaterals which are equiangular to one another and are such that the ratios *AB* to *EF*, and *BC* to *FG* are equal, the quadrilaterals are similar.

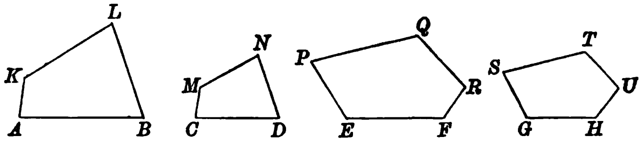
2. Prove that, if *ABCD*, *EFGH* be two quadrilaterals, which are equiangular to one another and are such that the ratios of *AB* to *EF* and *CD* to *GH* are equal, the quadrilaterals are similar.
What exceptional case may occur?

3. Prove that, if *ABCD*, *EFGH* be two quadrilaterals such that the angles *ABC*, *BCD* are equal to the angles *EFG*, *FGH* respectively, and the ratios of *AB* to *EF*, *BC* to *FG* and *CD* to *GH* are all equal, the quadrilaterals are similar.

PROPOSITION 22. PART 1.

If four straight lines be proportionals, the ratio of two similar polygons similarly described on the first pair is equal to the ratio of two similar polygons similarly described on the second pair.

Let the four straight lines *AB*, *CD*, *EF*, *GH* be proportionals, and let *AKLB*, *CMND* be two similar polygons similarly described on *AB*, *CD*, and *EPQRF*, *GSTUH* be two similar polygons similarly described on *EF*, *GH*:
it is required to prove that *AKLB* is to *CMND* as *EPQRF* to *GSTUH*.



PROOF. Because *AB* is to *CD* as *EF* to *GH*,
and *AKLB* has to *CMND* the ratio duplicate of the ratio of *AB* to *CD*, (Prop. 20, Coroll.)
and *EPQRF* has to *GSTUH* the ratio duplicate of the ratio of *EF* to *GH*,
therefore *AKLB* is to *CMND* as *EPQRF* to *GSTUH*. (V. Prop. 14, Coroll.)

Wherefore, *if four straight lines &c.*

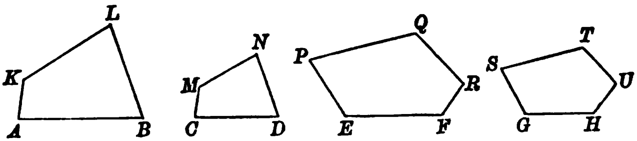
EXERCISE.

1. Perpendiculars are let fall from two opposite angles of a rectangle on a diagonal: shew that they will divide the diagonal into equal parts, if the square on one side of the rectangle be double that on the other.

PROPOSITION 22. PART 2.

If the ratio of two similar polygons similarly described on the first and the second of four straight lines be equal to the ratio of two similar polygons similarly described on the third and the fourth, the four straight lines are proportionals.

Let AB, CD, EF, GH be four given straight lines, and let $AKLB$, $CMND$ be two similar polygons similarly described on AB, CD, and $EPQRF$, $GSTUH$ be two similar polygons similarly described on EF, GH, and let $AKLB$, $CMND$, $EPQRF$, $GSTUH$ be proportionals:
it is required to prove that AB is to CD as EF to GH.



PROOF. Because $AKLB$ has to $CMND$ the ratio duplicate of the ratio of AB to CD, (Prop. 20, Coroll.)
and $EPQRF$ has to $GSTUH$ the ratio duplicate of the ratio of EF to GH,
and $AKLB$ is to $CMND$ as $EPQRF$ to $GSTUH$,
therefore AB is to CD as EF to GH. (V. Prop. 16.)

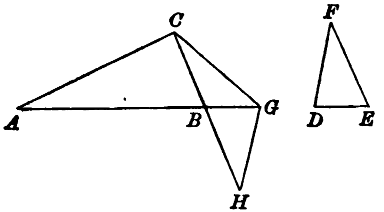
Wherefore, *if the ratio &c.*

PROPOSITION 23.

If two triangles have an angle of the one equal to an angle of the other, the ratio of the areas of the triangles is equal to the ratio compounded of the ratios of the sides about the equal angles.

Let the triangles *ABC*, *DEF* have the angles at *B* and *E* equal:
it is required to prove that the ratio of the triangle *ABC* to the triangle *DEF* is equal to the ratio compounded of the ratios *AB* to *DE* and *BC* to *EF*.

CONSTRUCTION. In *AB*, *CB* produced cut off *BG*, *BH* equal to *ED*, *EF*, and draw *CG*, *GH*.



PROOF. Because in the triangles *GBH*, *DEF*,
BG is equal to *ED*, and *BH* to *EF*,
and the angle *GBH* to the angle *DEF*,
the triangles are equal in all respects. (I. Prop. 4.)
And because the triangle *ABC* is to the triangle *BGC* as *AB* to *BG*, (Prop. 1.)
and the triangle *GCB* is to the triangle *GBH* as *CB* to *BH*;
therefore the triangle *ABC* has to the triangle *GBH* the ratio compounded of the ratios *AB* to *BG* and *CB* to *BH*; (V. Def. 8.)
therefore the triangle *ABC* has to the triangle *DEF* the ratio compounded of the ratios *AB* to *DE* and *BC* to *EF*.
Wherefore, *if two triangles* &c.

COROLLARY. *If two parallelograms have an angle of the one equal to an angle of the other, the ratio of the areas of the parallelograms is equal to the ratio compounded of the ratios of the sides about the equal angles.*

It is proved in Proposition 23 that the ratio of the triangle ABC to the triangle DEF is equal to the ratio compounded of the ratios AB to DE and BC to EF. Similarly it can be proved that the ratio of the triangle ABC to the triangle DEF is equal to the ratio compounded of the ratios BC to EF and AB to DE. And since any two ratios can be represented by the ratios AB to DE and BC to EF, if the lines be chosen of proper lengths, it follows that *the magnitude of the ratio compounded of two given ratios is independent of the order in which they are compounded.*

Again, because the proof of Proposition 23 is applicable to two right-angled triangles, we may assume the equal angles at B and E to be right angles, in which case the triangle ABC is equal to half the rectangle AB, BC and the triangle DEF is equal to half the rectangle DE, EF. It follows that the ratio compounded of AB to DE, and BC to EF is equal to the ratio of the rectangle AB, BC to the rectangle DE, EF, or, in other words, *the ratio compounded of the ratios of two pairs of lines is equal to the ratio of the rectangle contained by the antecedents to the rectangle contained by the consequents.*

EXERCISES.

1. A and B are two given points; AC and BD are perpendicular to a given straight line CD: AD and BC intersect at E, and EF is perpendicular to CD: shew that AF and BF make equal angles with CD.

2. If a triangle inscribed in another have one side parallel to a side of the other, its area is to that of the larger triangle as the rectangle contained by the segments of either of the other sides of the original triangle is to the square on that side.

3. If on two straight lines $OABC$, $OFED$, the points be so chosen that AE is parallel to BD, and AF parallel to CD, then also BF is parallel to CE.

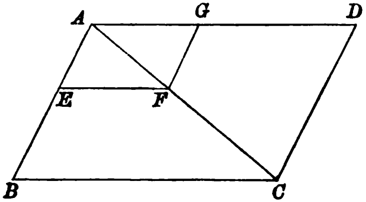
4. Find the greatest triangle which can be inscribed in a given triangle so as to have one side parallel to one of the sides of the given triangle.

5. Find the least triangle which can be described about a given triangle.

PROPOSITION 24.

A parallelogram about a diagonal of another parallelogram is similar to it.

Let the parallelogram $AEFG$ be about the diagonal AC of the parallelogram $ABCD$:
it is required to prove that $AEFG$ is similar to $ABCD$.



PROOF. Because EF is parallel to BC,
the angles AEF, AFE are equal to the angles ABC, ACB respectively, (I. Prop. 29.)
and therefore the triangles AEF, ABC are equiangular to one another;
therefore the parallelograms $AEFG$, $ABCD$ are equiangular to one another.

And because the triangles AEF, ABC are equiangular to one another,

AE is to AB as EF to BC; (Prop. 4.)
and EF is equal to AG, and BC to AD; (I. Prop. 34.)
therefore also AE is to AB as AG to AD.

Similarly it can be proved that the ratios of all pairs of corresponding sides of the parallelograms $AEFG$, $ABCD$ are equal.

Therefore the parallelograms are similar.

Wherefore, *a parallelogram* &c.

EXERCISES.

1. Prove that, in the figure of VI. 24, *EG* and *BD* are parallel.

2. Prove that, if in the figure of VI. 24, *EF*, *GF* produced cut *CD*, *CB* in *H*, *K*, then *HG*, *CA*, *KE* meet in a point.

3. Prove that, if two similar quadrilaterals *ABCD*, *AEFG* be so placed that *ABE*, *ADG* are straight lines, then the points *A*, *F*, *C* lie on a straight line.

4. In a given triangle inscribe a rhombus which shall have one of its angular points at a given point in the base, and a side on that base.

5. Construct a parallelogram similar to a given parallelogram, so that two of its vertices are on one side of a given triangle and the other vertices on the other two sides.

PROPOSITION 25.

To construct a polygon similar to a given polygon and equal to another given polygon.

Let $ABCDE$ be one given polygon, and $FGHK$ another: it is required to construct a polygon similar to $ABCDE$ and equal to $FGHK$.

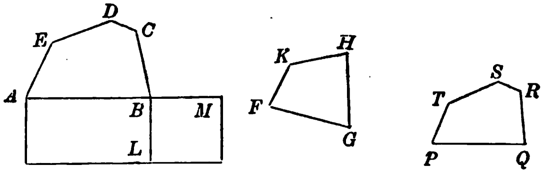
CONSTRUCTION. Construct on AB a rectangle AL equal to $ABCDE$,

and on BL construct a rectangle LM equal to $FGHK$. (I. Prop. 45.)

Find PQ a mean proportional between AB and BM, (Prop. 13.)

and on PQ construct a polygon $PQRST$ similar to $ABCDE$, so that PQ, AB are corresponding sides: (Prop. 18.)

then $PQRST$ is a polygon constructed as required.



PROOF. Because $ABCDE$ is to $PQRST$ in the ratio duplicate of the ratio of AB to PQ,

and AB is to BM in the ratio duplicate of the ratio of AB to PQ;

therefore $ABCDE$ is to $PQRST$ as AB to BM;

and AB is to BM as the rectangle AL to the rectangle LM,

that is, as $ABCDE$ to $FGHK$.

Therefore $PQRST$ is equal to $FGHK$;

and it was constructed similar to $ABCDE$.

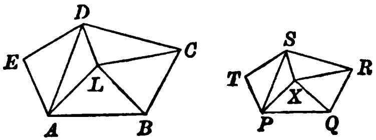
Wherefore, *a polygon PQRST has been constructed similar to the polygon ABCDE and equal to the polygon FGHK.*

EXERCISES.

1. Construct a square equal to a given equilateral triangle.
2. Construct an equilateral triangle equal to a given rectangle.

In Proposition 4 it was proved that, *if two triangles be equiangular to one another, they are similar*. Hence the condition of the equality of the ratios of corresponding sides, which appears in Definition 2, is unnecessary in the case of two *triangles*, which are equiangular to one another.

If we take the case of two polygons $ABCDE$, $PQRST$ of more than three sides, which are equiangular to one another, and which are such that all but two of the ratios of pairs of corresponding sides are equal, say AB to PQ, BC to QR, CD to RS, where two adjacent sides are omitted, we can prove that the polygons are similar.



Take any point L within $ABCD$, and draw LA, LB, LC, LD, DA. Within the polygon $PQRS$ draw PX, QX, making the angles QPX, PQX equal to the angles BAL, ABL respectively, and draw XR, XS, SP.

It can be proved, as in Proposition 20, that the triangles ALB, BLC, CLD are similar to the triangles PXQ, QXR, RXS respectively; therefore the angles ALB, BLC, CLD are equal to the angles
PXQ, QXR, RXS respectively, and
therefore the angle ALD is equal to the angle PXS;
also each of the ratios LA to XP, LB to XQ, LC to XR and LD to XS
is equal to the ratio of AB to PQ, and
therefore AL is to PX as LD to XS.

Therefore the triangles ALD, PXS are similar, (Prop. 6.)
and AD is to PS as LA to XP.

Hence the two triangles AED, PTS are equiangular to one another; therefore they are similar, and each of the ratios DE to ST, EA to TP is equal to the ratio of AD to PS, (Prop. 4) which is equal to the ratio of LA to XP, and therefore to the ratio AB to PQ.

In this case therefore the two polygons are similar.

This method reduces the case, where the two sides whose ratios are omitted are adjacent, to the similar case of quadrilaterals (Ex. 1, page 395). A similar method will reduce the case, where the two sides whose ratios are omitted are not adjacent, to the similar case of quadrilaterals (Ex. 2, page 395).

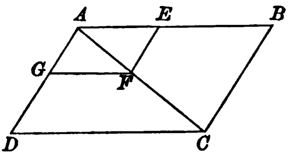
The two cases together justify the remark on Definition 2, page 350.

PROPOSITION 26.

If two similar parallelograms have a common angle and be similarly placed, one is about the diagonal of the other.

Let the parallelograms $ABCD$, $AEFG$ be similar and similarly placed and have a common angle at A:
it is required to prove that the points A, F, C lie on a straight line.

CONSTRUCTION. Draw AF and AC.



PROOF. Because the parallelograms $AEFG$, $ABCD$ are similar,

AG is to AD as GF to DC; (Def. 2.)
and the angle AGF is equal to the angle ADC;
therefore the triangles AGF, ADC are equiangular to one another. (Prop. 6.)
Therefore the angle GAF is equal to the angle DAC,
i.e. the three points A, F, C lie on a straight line.

Wherefore, *if two similar parallelograms* &c.

EXERCISE.

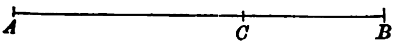
1. Inscribe in a given triangle a parallelogram similar to a given parallelogram so as to have two corners on one side and one on each of the other sides of the triangle.

PROPOSITION 30.

To divide a given straight line in extreme and mean ratio.

Let AB be the given straight line:
it is required to divide it in extreme and mean ratio.

CONSTRUCTION. Divide AB at the point C into two parts so that the rectangle AB, BC may be equal to the square on AC. (II. Prop. 11.)



PROOF. Because the rectangle AB, BC is equal to the square on AC,

AB is to AC as AC to BC. (Prop. 17.)

Wherefore, *the given straight line AB has been divided at C in extreme and mean ratio.*

EXERCISES.

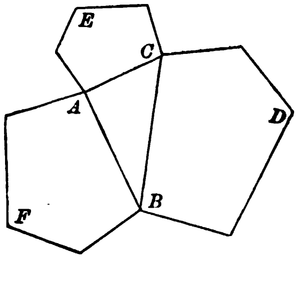
1. Two diagonals of a regular pentagon which meet within the figure divide each other in extreme and mean ratio.

2. Divide a given straight line into two parts so that any triangle described on the first part may have to a similar and similarly described triangle on the second part the ratio which the whole has to the second part.

PROPOSITION 31.

A polygon on the hypotenuse of a right-angled triangle is equal to the sum of the polygons similarly described on the other sides.

Let ABC be a right-angled triangle having the right angle BAC:
and let BDC, CEA, AFB be similar polygons similarly described on BC, CA, AB respectively:
it is required to prove that the polygon BDC is equal to the sum of the polygons CEA, AFB.



Proof. Because BDC has to CEA the ratio duplicate of the ratio of BC to CA,
and the square on BC has to the square on CA the ratio duplicate of the ratio of BC to CA; (Prop. 20, Coroll.)
therefore BDC is to CEA as the square on BC to the square on CA;
therefore BDC is to the square on BC as CEA to the square on CA. (V. Prop. 9.)

Similarly it can be proved that

BDC is to the square on *BC* as *AFB* to the square on *AB*.

Therefore *BDC* is to the square on *BC*

as the sum of *CEA*, *AFB* to the sum of the squares on *CA*, *AB*; (V. Prop. 6.)

and the square on *BC* is equal to the sum of the squares on *CA*, *AB*; (I. Prop. 47.)

therefore *BDC* is equal to the sum of *CEA*, *AFB*.

Wherefore, *a polygon* &c.

EXERCISES.

1. Divide a given finite straight line into two parts so that the squares on them shall be to one another in a given ratio.

2. Construct an equilateral triangle equal to the sum of two given equilateral triangles.

3. On two given lines similar triangles are described; construct a similar triangle equal to the difference of the given triangles.

4. Construct a triangle equal to the sum of three given similar triangles and similar to them.

5. Construct a polygon equal to the sum of any number of similar polygons and similar to them.

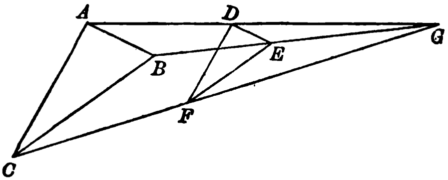
PROPOSITION 32.

If two triangles have sides parallel in pairs, the straight lines joining the corresponding vertices meet in a point.

Let ABC, DEF be two triangles such that the sides BC, CA, AB are parallel to the sides EF, FD, DE respectively:

it is required to prove that the straight lines joining the pairs of points A, D; B, E; C, F meet in a point.

CONSTRUCTION. Draw AD, BE and let them meet in G; and draw GC, GF.



PROOF. Because AB is parallel to DE,

the angles GAB, GBA are equal to the angles GDE, GED respectively; (I. Prop. 29.)

therefore the triangles GAB, GDE are equiangular to one another; (I. Prop. 32.)

therefore GB is to GE as BA to ED; (Prop. 4.)

and because the triangles ABC, DEF are equiangular to one another, (I. Prop. 34, Coroll. 2.)

BA is to ED as BC to EF; (Prop. 4.)

therefore GB is to GE as BC to EF; (V. Prop. 5.)

and the angle GBC is equal to the angle GEF; (I. Prop. 29.)

therefore the triangles GBC, GEF are equiangular to one another; (Prop. 6.)

therefore the angles BGC, EGF are equal,

that is, the points C, F, G lie on a straight line,

or, in other words, AD, BE, CF meet in a point.

Wherefore, *if two triangles* &c.

It will be seen at once that, if in the diagram of Proposition 32 AB be equal to DE, then the straight lines AD, BE do not meet at any point at a finite distance, in other words, they are parallel. Also, because the triangles ABC, DEF are similar, if AB be equal to DE, then also BC is equal to EF, and therefore BE and CF are parallel.

Hence we must consider the case when the two triangles are similarly placed and equal as a special case in which the point of intersection of the lines joining the corresponding vertices is at an infinite distance.

EXERCISES.

1. If two similar triangles be similarly placed on two parallel straight lines, the lines joining corresponding vertices meet in a point.

2. If any two similar polygons have three pairs of corresponding sides parallel, the straight lines joining the corresponding vertices meet in a point.

3. AB is a fixed diameter of a circle ABC: PQ is a straight line parallel to AB and of constant length, which moves so that its middle point traces out the circle ABC; find the locus of the intersection of AP, BQ and of AQ, BP.

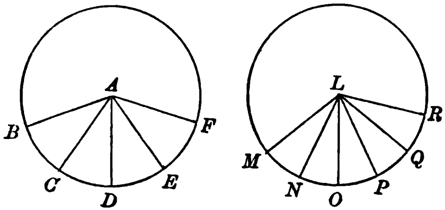
4. Prove that, if the corresponding sides of $ABCD$, $EFGH$ two squares be parallel, the straight lines AE, BF, CG, DH pass through a point, and AG, BH, CE, DF pass through another point.

PROPOSITION 33. PART 1.

In equal circles angles at the centres have the same ratio as the arcs on which they stand.

Let BCD, MNO be two given equal circles, and let BAC, MLN be two angles at their centres:
it is required to prove that the angle BAC is to the angle MLN as the arc BC to the arc MN.

CONSTRUCTION. From A draw any number of radii AD, AE, AF making the angles CAD, DAE, EAF each equal to the angle BAC; and from L draw any number of radii LO, LP, LQ, LR making the angles NLO, OLP, PLQ, QLR each equal to the angle MLN.



PROOF. Because the angles BAC, CAD, DAE, EAF are all equal,

the arcs BC, CD, DE, EF are all equal; (III. Prop. 26.)
therefore the angle BAF and the arc BF are equimultiples of the angle BAC and the arc BC.

Similarly it can be proved that
the angle MLR and the arc MR are equimultiples of the angle MLN and the arc MN.

And, because the circles are equal, if the angle BAF be equal to the angle MLR,

the arc BF is equal to the arc MR, (III. Prop. 26.)
and if the angle BAF be greater or less than the angle MLR, the arc BF is greater or less respectively than the arc MR.
Therefore the angle BAC is to the angle MLN as the arc BC to the arc MN. (V. Def. 5.)

Wherefore, *in equal circles* &c.

COROLLARY. *In equal circles angles at the circumferences have the same ratio as the arcs on which they stand.*

The angles at the centres are double of the angles at the circumferences, and therefore have the same ratio.
(V. Prop. 6, Coroll.)

In the construction of Proposition 33 there is nothing to limit the magnitude of the multiple angles *BAF*, *MLR*; they may be greater than two right angles, greater than four right angles, or greater than any multiple of four right angles, and at the same time the multiple arcs *BF*, *MR* will be greater than half the circumference of the circle, greater than the circumference, or greater than any multiple of the circumference.

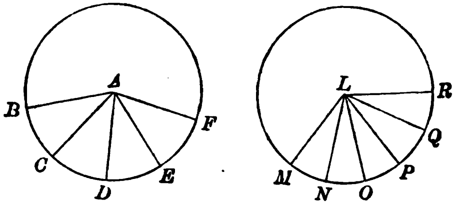
In the Third Book (page 221) we had occasion to remark that the admittance of angles equal to or greater than two right angles was not inconsistent with Euclid's methods. We may now go further and say that the admittance of angles without any restriction whatever on their magnitude is essential to his method. The validity of the proof of this Proposition depends on the possibility of choosing any multiples we please of the angles *BAC*, *MLN*, that is, of taking the multiple angles *BAF*, *MLR* as large as we please.

PROPOSITION 33. PART 2.

In equal circles, the areas of sectors have the same ratio as their angles.

Let *BCD*, *MNO* be two given equal circles, and let *BAC*, *MLN* be two angles at their centres:
it is required to prove that the angle *BAC* is to the angle *MLN* as the sector *BAC* to the sector *MLN*.

CONSTRUCTION. From *A* draw any number of radii *AD*, *AE*, *AF* making the angles *CAD*, *DAE*, *EAF* each equal to the angle *BAC*; and from *L* draw any number of radii *LO*, *LP*, *LQ*, *LR* making the angles *NLO*, *OLP*, *PLQ*, *QLR* each equal to the angle *MLN*.



PROOF. Because the angle *CAD* is equal to the angle *BAC*, it is possible to shift the figure *CAD* so that *AC* will be on *AB*, and *AD* on *AC*; if this be done, then the point *C* will coincide with *B* and *D* with *C*,
and therefore the arc *CD* with the arc *BC*. (III. Prop. 23.)
Therefore the sector *CAD* coincides with the sector *BAC* and is equal to it in all respects.

Similarly it can be proved that the sectors *DAE*, *EAF* are each equal to the sector *BAC*.
Therefore the angle *BAF* and the sector *BAF* are equimultiples of the angle *BAC* and the sector *BAC*.

Similarly it can be proved that
the angle *MLR* and the sector *MLR* are equimultiples of the angle *MLN* and the sector *MLN*.

And it can be proved as before that, if the angle BAF be equal to the angle MLR,

the sector BAF is equal to the sector MLR;

and, if the angle BAF be greater or less than the angle MLR, the sector is greater or less respectively than the sector MLR; therefore the angle BAC is to the angle MLN as the sector BAC to the sector MLN. (V. Def. 5.)

Wherefore, *in equal circles* &c.

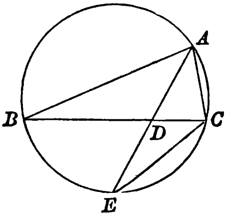
COROLLARY. *In equal circles the areas of sectors have the same ratio as the arcs on which they stand.*

PROPOSITION 34.

If an angle of a triangle be bisected by a straight line which cuts the opposite side, the rectangle contained by the segments of that side is less than the rectangle contained by the other sides by the square on the line.

Let the angle BAC of the triangle ABC be bisected by the straight line AD, which cuts BC at D:
it is required to prove that the rectangle BD, DC is less than the rectangle BA, AC by the square on AD.

CONSTRUCTION. Describe the circle ABC; (IV. Prop. 5.)
produce AD to meet the circle at E, and draw EC.



PROOF. Because in the triangles BAD, EAC,
the angle BAD is equal to the angle EAC, (Hypothesis.)
and the angle ABD to the angle AEC, (III. Prop. 21.)
therefore the triangles are equiangular to one another;
therefore BA is to EA as AD to AC; (Prop. 4.)
therefore the rectangle BA, AC is equal to the rectangle EA, AD, (Prop. 16.)
that is, to the rectangle ED, DA together with the square on AD. (II. Prop. 3.)
And the rectangle ED, DA is equal to the rectangle BD, DC; (III. Prop. 35.)
therefore the rectangle BD, DC is less than the rectangle BA, AC by the square on AD.

Wherefore, *if an angle* &c.

EXERCISES.

1. If an angle of a triangle be bisected externally by a straight line which cuts the opposite side produced, the rectangle contained by the segments of that side is greater than the rectangle contained by the other sides by the square on the line.

2. Prove that, if the internal and the external bisectors of the vertical angle of a triangle ABC cut BC in D and E, then the square on DE is equal to the difference of the rectangles EB, EC and DB, DC.

3. If I be the centre of the inscribed circle of a triangle ABC and AI produced cut the circumscribed circle ABC in E, then the rectangle contained by AI, IE is equal to twice the rectangle contained by the radii of the circumscribed and the inscribed circles.
(See Ex. 46, page 324.)

4. If I_1 be the centre of the circle of the triangle ABC escribed beyond BC and AI_1 cut the circumscribed circle ABC in E, then the rectangle contained by AI_1, I_1E is equal to twice the rectangle contained by the radii of the circumscribed and the escribed circles.

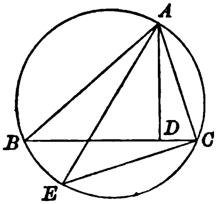
5. AB is the base of a triangle ABC whose sides are segments of a line divided in extreme and mean ratio. CP the bisector of the angle C, and CQ the perpendicular from C on AB meet AB in P and Q. Prove that the square on CP is equal to twice the rectangle contained by PQ and AB.

PROPOSITION 35.

If a perpendicular be drawn from a vertex of a triangle to the opposite side, the rectangle contained by the other sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.

Let AD be the perpendicular drawn from the vertex A of the given triangle ABC to the opposite side BC:
it is required to prove that the rectangle contained by AB, AC is equal to the rectangle contained by AD and the diameter of the circle described about ABC.

CONSTRUCTION. Describe the circle ABC; (IV. Prop. 5.)
draw the diameter AE and draw EC.



PROOF. Because in the triangles BAD, EAC,
the angle ABD is equal to the angle AEC, (III. Prop. 21.)
and the angle ADB to the angle ACE; (III. Prop. 31.)
therefore the triangles are equiangular to one another, (I. Prop. 32.)
and BA is to EA as AD to AC; (Prop. 4.)
therefore the rectangle BA, AC is equal to the rectangle EA, AD. (Prop. 16.)

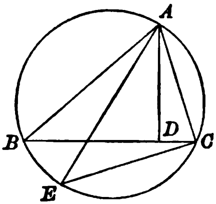
Wherefore, *if a perpendicular* &c.

PROPOSITION 35 A.

The ratio of twice the area of a triangle to the rectangle contained by two of the sides is equal to the ratio of the third side to the diameter of the circumscribed circle of the triangle.

Let ABC be a triangle:
it is required to prove that twice the area of ABC is to the rectangle contained by AC, BC as AB to the diameter of the circle described about ABC.

CONSTRUCTION. Describe the circle ABC; draw the diameter AE, draw AD perpendicular to BC and draw EC.



PROOF. Because in the triangles BAD, EAC,
the angle ABD is equal to the angle AEC, (III. Prop. 21.)
and the angle ADB to the angle ACE; (III. Prop. 31.)
therefore the triangles are equiangular to one another, (I. Prop. 32.)
and AD is to AC as AB to AE. (Prop. 4.)
Therefore the rectangle AD, BC is to the rectangle AC, BC as AB to AE; (Prop. 1.)
and the rectangle AD, BC is equal to twice the area of the triangle ABC;
therefore twice the area of the triangle ABC is to the rectangle AC, BC as AB to the diameter of the circle ABC.

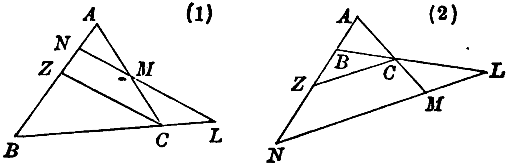
Wherefore, *the ratio* &c.

ADDITIONAL PROPOSITION 1.

*If a straight line cut the three sides of a triangle produced if necessary, the ratio compounded of the ratios of the segments of the sides taken in order is equal to unity.**

Let the sides BC, CA, AB of the triangle ABC be cut by the straight line LMN in L, M, N respectively.

Through C draw CZ parallel to LMN to meet ABN in Z.



Because ZC, NML are parallel,

AM is to MC as AN to NZ,

and CL is to LB as ZN to NB; (Prop. 2.)

therefore the ratio compounded of the ratios AM to MC and CL to LB is equal to the ratio compounded of the ratios AN to NZ and ZN to NB,

i.e. the ratio AN to NB; (V. Def. 8.)

therefore the ratio compounded of the ratios AM to MC, CL to LB and BN to NA is equal the ratio compounded of the ratios AN to NB and NB to AN, that is, to the ratio AN to AN, i.e. to unity. (V. Def. 2.)

* This theorem is attributed to Menelaus, a Greek Geometer, who lived in the latter part of the first century A.D.

Definition. *A straight line drawn to cut a series of lines is often called* **a transversal.**

The straight line LMN in Additional Proposition 1 is **a transversal** of the triangle ABC.

EXERCISES.

1. Points E, F are taken in the sides AC, AB of a triangle such that AE is twice EC and BF is twice FA; FE produced cuts BC in D; find the ratio BD to DC.

2. If the bisectors of the angles B, C of a triangle ABC meet the opposite sides in D and E, and if the straight line DE produced meet BC produced in F, then the external angle at A is bisected by AF.

3. BD is the perpendicular let fall from one end of the base upon the straight line bisecting the vertical angle BAC of a triangle. If BA be three times as long as AC, AD will be bisected at the point E, where it cuts the base.

4. If a side BC of a triangle ABC be bisected by a straight line which meets the sides AB, AC, produced if necessary, in D and E respectively, then AE is to EC as AD to DB.

5. If one side of a given triangle be produced and the other shortened by equal quantities, the line joining the points of section will be divided by the base in the inverse ratio of the sides.

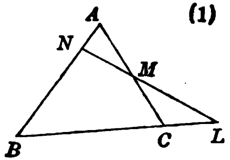
6. In the sides AB, AC of a triangle ABC two points D, E are taken such that BD is equal to CE; DE, BC are produced to meet at F: shew that AB is to AC as EF to DF.

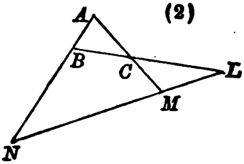
The converse of the theorem on page 418 may be stated as follows:—

ADDITIONAL PROPOSITION 2.

If three points be taken on the sides of a triangle (either one on a side produced and the other two on sides, or else all three on sides produced), such that the ratio compounded of the ratios of the segments of the sides taken in order is equal to unity, the three points lie on a straight line.

Let three points L, M, N be taken on the sides BC, CA, AB of a triangle ABC, either all on sides produced (fig. 2) or one L on a side produced, and the others M, N on sides (fig. 1) such that the ratio compounded of the ratios AM to MC, CL to LB and BN to NA is equal to unity.





Draw LM and let it produced cut AB in P.

Then the ratio compounded of the ratios

AM to MC, CL to LB and BP to PA is equal to unity;

(Add. Prop. 1.)

and the ratio compounded of the ratios

AM to MC, CL to LB and BN to NA is equal to unity;

(Hypothesis.)

therefore the ratio BP to PA is equal to the ratio BN to NA;

therefore BP is to BA as BN to BA; (V. Prop. 10 or 11)

therefore BP is equal to BN; (V. Prop. 3.)

that is, P coincides with N,

or, in other words, L, M, N are in a straight line.

EXERCISES.

1. The inscribed circle of a triangle *ABC* touches the sides *BC*, *CA*, *AB* at *D*, *E*, *F*; *EF*, *FD*, *DE* produced meet *BC*, *CA*, *AB* in *L*, *M*, *N*: prove that *L*, *M*, *N* are collinear.

2. An escribed circle of a triangle *ABC* touches the side *BC* at *D* and the sides *AC*, *AB* produced at *E*, *F*; *ED*, *FD* produced cut *AB*, *AC* in *K*, *H* respectively; prove that *FE*, *BC*, *KH* meet in a point.

3. If *AB*, *CD*, *EF* be three parallel straight lines, and *AC*, *BD* meet in *N*, *CE*, *DF* meet in *L*, and *EA*, *FB* meet in *M*, then *L*, *M*, *N* lie on a straight line.

4. If *D*, *E*, *F* be the points of contact with *BC*, *CA*, *AB* of the inscribed circle, or of any one of the escribed circles of the triangle *ABC*, the lines *AD*, *BE*, *CF* pass through a point.

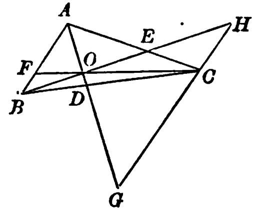
5. If *D* be the point of contact of the inscribed circle of a triangle *ABC* with *BC*, and *E*, *F* the points of contact of escribed circles with *CA* produced and *BA* produced respectively, then *AD*, *BE*, *CF* meet in a point.

6. If one escribed circle of a triangle *ABC* touch *AC* in *F* and *BA* produced in *G* and another escribed circle touch *AB* in *H* and *CA* produced in *K*, then *FH*, *KG* produced cut *BC* produced in points equidistant from the middle point of *BC*.

ADDITIONAL PROPOSITION 3.

*If three straight lines be drawn from the vertices of a triangle meeting in a point and cutting the opposite sides or the sides produced, the ratio compounded of the ratio of the segments of the sides taken in order is equal to unity.**

Let the straight lines AO, BO, CO be drawn from the vertices of the triangle ABC meeting in O, and cutting BC, CA, AB in D, E, F respectively.



Through C draw HCG parallel to AB to meet BO, AO produced in H, G.

Then because the triangles AOF, GOC are equiangular to one another,

AF is to GC as FO to CO; (Prop. 4.)

and because the triangles FOB, COH are equiangular to one another,

FB is to CH as FO to CO;

therefore AF is to GC as FB to CH, (V. Prop. 5.)

and therefore AF is to FB as GC to CH. (V. Prop. 9.)

And because the triangles CEH, AEB are equiangular to one another,

CE is to AE as CH to AB;

and because the triangles BDA, CDG are equiangular to one another,

BD is to CD as BA to CG;

therefore the ratio compounded of the ratios

AF to FB, CE to EA, and BD to DC

is equal to the ratio compounded of the ratios

GC to CH, CH to AB, and AB to CG,

which is equal to unity.

* This theorem was first published in the year 1678 by Giovanni Ceva, an Italian.

In Additional Proposition 3 it has been proved that, if through the vertices A, B, C of a triangle three concurrent straight lines AD, BE, CF be drawn meeting the sides BC, CA, AB in D, E, F, the ratio of BD to DC is equal to the ratio compounded of the ratios BF to FA and AE to EC.

In Additional Proposition 1 it has been proved that, if the straight line FE be drawn and produced to meet BC produced in L, the ratio BL to LC is equal to the ratio compounded of the ratios BF to FA and AE to EC.

Therefore BD is to DC as BL to LC,

or, in other words, $BDCL$ is **a harmonic range.**

It is a remarkable fact that, although the theorem on page 418 had been known as early as the 1st century, the theorem on page 422, which seems a very natural complement to the other, should not have been discovered until the 17th century.

EXERCISES.

1. Prove Ceva's Theorem for a triangle ABC and a point O, (1) when O lies between AB produced and AC produced, (2) when O lies between BA produced and CA produced.

2. Prove Ceva's Theorem by using the result of Ex. 3, page 355.

3. Prove Ceva's Theorem by the use of Menelaus' Theorem, considering in the figure of Add. Prop. 3 COF a transversal of the triangle ABD and BOE a transversal of the triangle ADC.

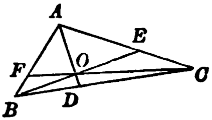
4. D, E, F are the points in which the bisectors of the angles A, B, C of a triangle cut the opposite sides; prove that, if BC be equal to half the sum of the sides AB, AC, then EF bisects AD.

The converse of the theorem on page 422 may be stated as follows;—

ADDITIONAL PROPOSITION 4.

If three straight lines be drawn through the vertices of a triangle cutting the opposite sides (either all three sides, or else one side and the other two sides produced) so that the ratio compounded of the ratios of the segments of the sides taken in order is equal to unity, the three straight lines meet in a point.

Let three straight lines *AD*, *BE*, *CF* be drawn from the vertices *A*, *B*, *C* of a triangle *ABC* to cut the opposite sides in *D*, *E*, *F* respectively, so that the ratio compounded of the ratios *AF* to *FB*, *BD* to *DC* and *CE* to *EA* is equal to unity.



Let *AD*, *CF* meet in *O*: draw *BO* and produce it to meet *CA* in *P*.

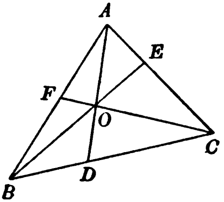
Then because the ratio compounded of the ratios *AF* to *FB*, *BD* to *DC* and *CP* to *PA* is equal to unity, (Add. Prop. 3.)
and also the ratio compounded of the ratios
AF to *FB*, *BD* to *DC* and *CE* to *EA* is equal to unity; (Hypothesis.)
therefore the ratio *CP* to *PA* is equal to the ratio *CE* to *EA*;
therefore *CP* is to *CA* as *CE* to *CA*; (V. Prop. 10 or 11.)
therefore *CP* is equal to *CE*; (V. Prop. 8.)
therefore *P* coincides with *E*,
or, in other words, *AD*, *BE*, *CF* meet in a point.

If in the sides BC, CA, AB of a triangle, points D, E, F be taken such that BD is to DC as n to m, CE is to EA as l to n, and AF is to FB as m to l, where l, m, n are any three integers,
the straight lines AD, BE, CF meet in a point, say O. (Add. Prop. 4.)

It is proved by Add. Prop. 1 that the ratio of AO to OD is equal to the ratio compounded of the ratios AE to EC and CB to BD, that is, of the ratios n to l and $m+n$ to n;

therefore AO is to OD as $m+n$ to l.

Similarly it appears that BO is to OE as $n+l$ to m and that CO is to OF as $l+m$ to n.



It follows that, if we divide BC in D so that BD is to DC as n to m, and then divide DA in O so that DO is to OA as l to $m+n$, we arrive at the same point, as if we divide CA in E so that CE is to EA as l to n, and then divide EB in O so that EO is to OB as m to $n+l$, or as if we divide AB in F so that AF is to FB as m to l, and then divide FC in O so that FO is to OC as n to $l+m$.

This point O is called **the centroid of weights** l, m, n at A, B, C respectively. It appears that the position of the centroid of three weights is independent of the order in which the weights are taken, or, in other words, the centroid of three weights is a unique point.

It is not difficult to see that this proposition can be extended to any number of weights, so that we may state the proposition in the general form, *the centroid of a number of given weights is a unique point.*

EXERCISE.

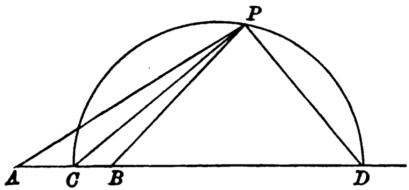
1. From the vertex A of a triangle ABC a straight line is drawn cutting BC in D, and the angles BDA, CDA are bisected by straight lines cutting AB, AC in F, E respectively: prove that AD, BE, CF intersect in a point.

ADDITIONAL PROPOSITION 5.

*The locus of a point, the ratio of whose distances from two given points is constant, is a circle**.

Let A, B be two given points and P a point such that the ratio of AP to BP is equal to the given ratio.

Draw PA, PB; and draw PC, PD the internal and the external bisectors of the angle APB meeting AB in C and AB produced in D.



Because PC, PD are the bisectors of the angle APB,
therefore the ratios of AC to CB and AD to DB are equal to the ratio of AP to PB; and the ratio of AP to PB is constant;
therefore C and D are two fixed points. (Ex. 1, page 359.)
And because PC, PD are the bisectors of the angle APB,
the angle CPD is a right angle. (Ex. 5, page 43.)
Therefore every point on the locus of P must lie on the circle upon the fixed line CD as diameter.

Next we will prove that every point of the circle belongs to the locus.

Let P be any point on the circle described on CD as diameter.

Draw PA, PC; and draw PE making the angle CPE equal to the angle CPA and meeting CD at E;
then AP is to PE as AC to CE. (Prop. 3, Part 1.)
Again, because CPD is a right angle,
PD is the external bisector of the angle APE;
therefore AP is to PE as AD to DE.

* This theorem is attributed to Apollonius of Perga, a Greek geometer, who lived in the latter part of the third century B.C.

Therefore AC is to CE as AD to DE; (V. Prop. 5.)

and AC is to CB as AD to DB;

therefore CE is to ED as CB to BD,

and E coincides with B, which is a fixed point. (Ex. 1, page 359.)

Therefore AP is to PB in the fixed ratio AC to CB for every point P on the circle.

We may state the result of this proposition thus:—*If a circle be described upon the straight line joining two conjugate points of a harmonic range as a diameter, the ratio of the distances of a point on the circle from the other pair of conjugate points is constant.*

EXERCISES.

1. Prove that, if A, B be two fixed points and P be a point such that PA is equal to m times PB,

(1) if m vanish, the locus reduces to the point A;

(2) if m be equal to unity, the locus is the straight line which bisects AB at right angles;

(3) if m be infinitely great, the locus reduces to the point B;

(4) if m be greater than unity, the locus is a circle excluding A and including B;

(5) if m be less than unity, the locus is a circle including A and excluding B;

(6) if m be greater than unity, the greater the value of m, the less the circle;

(7) if m be less than unity, the less the value of m, the less the circle;

(8) the loci for two different values of m do not intersect.

2. A and B are the centres of two circles. A straight line PQ parallel to AB meets the circles in P and Q: find the locus of the point of intersection of AP and BQ.

3. Find a point such that its distances from three given points may be in given ratios.

4. Prove that, if a map be laid flat on another map of the same district on a larger scale, there is one place in the district which is represented in the two maps by points which are superposed one on the other.

ADDITIONAL PROPOSITION 6.

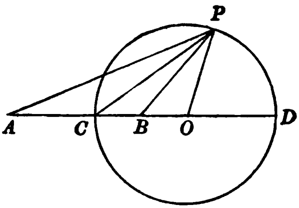
If the line between one pair of conjugates of a harmonic range be bisected, the square on half the line is equal to the rectangle contained by the segments of the line between the other pair of conjugates made by the point of bisection.

Let *ACBD* be a harmonic range, such that *AC* is to *CB* as *AD* to *DB*, so that *A*, *B* are one pair of conjugates and *C*, *D* the other.

First, let *O* be the middle point of *CD*, the line between one pair of conjugates.

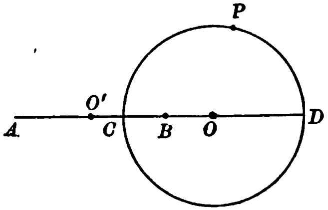
Describe the circle on *CD* as diameter.

Take any point *P* on the circle, and draw *PA*, *PC*, *PB*, *PO*.



Because the angle *OPC* is equal to the angle *OCP*,
the sum of the angles *OPB*, *BPC* is equal to the sum of the angles *CAP*, *CPA*; (I. Prop. 32.)
and because *AP* is to *PB* as *AC* to *CB*, (see page 427.)
the angle *BPC* is equal to the angle *CPA*;
therefore the angle *OPB* is equal to the angle *OAP*;
therefore *OP* touches the circle described about *APB*; (Converse of III. Prop. 32.)
therefore the square on *OP*, which is equal to the square on *OC*, is equal to the rectangle *OA*, *OB*. (III. Prop. 36, Coroll.)

Secondly, let O' be the middle point of AB, the line between the second pair of conjugates.



The rectangle OA, OB is equal to the difference between the squares on OO' and $O'B$, (II. Prop. 10.)

and the rectangle OA, OB is equal to the square on OC;

therefore the square on OC is equal to the difference between the squares on OO' and $O'B$.

Therefore the square on $O'B$ is equal to the difference between the squares on OO' and OC, which is equal to the rectangle $O'C$, $O'D$. (II. Prop. 10.)

NOTE. All the circles, which are the loci of the point P for different values of the ratio AP to BP, have their centres in the line AB, and, since the rectangle $O'C$, $O'D$ is equal to the square on the tangent from O' to the circle CPD, the straight line which bisects AB at right angles is the radical axis of *every pair* of such circles. Such a series of circles is called **Coaxial**.

The points A, B are called **the limiting points** of the series of circles.

EXERCISES.

1. If two circles be described upon the straight lines joining the two pairs of conjugate points of a harmonic range as diameters, the circles cut orthogonally.

2. A common tangent to two given circles is divided harmonically by any circle which is coaxial with the given circles.

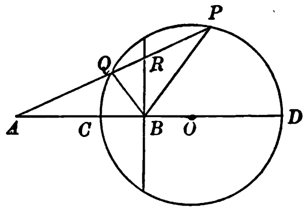
ADDITIONAL PROPOSITION 7.

A chord of a circle is divided harmonically by any point on it and the polar of the point.

Let PQ be a chord of the given circle $CQPD$: take A any point on PQ produced and draw the diameter ACD.

Let B be the point such that $ACBD$ is a harmonic range.

Draw PB, BQ and draw BR at right angles to AB meeting PQ in R.



Because $ACBD$ is a harmonic range and O is the middle point of CD, the rectangle OA, OB is equal to the square on OC; (Add. Prop. 6.)

therefore BR is the polar of A. (see page 259.)

Because AP is to PB as AC to CB,

and AQ is to QB as AC to CB; (Add. Prop. 5.)

therefore AP is to PB as AQ to QB;

and PB is to PA as QB to AQ; (V. Def. 5 note.)

therefore PB is to BQ as PA to AQ; (V. Prop. 9.)

therefore BA is the external bisector of the angle PBQ; (Prop. 3, Part 2.)

therefore BR is the internal bisector,

and PR is to RQ as PB to BQ, (Prop. 3, Part 1.)

and therefore as PA to AQ;

therefore $AQRP$ is a harmonic range.

It may be remarked that in the diagram on page 430 the point A is taken outside the circle. If the point were inside the circle, say R, its polar would intersect PQR in A. (Add. Prop. on page 262.)

Hence the theorem is established generally.

EXERCISES.

1. Prove that, if $ACBD$ be a harmonic range, and if O be the middle point of CD, then AC is to CB as AO to OC.

2. Establish the theorem of page 426 by proving that in the figure of page 430 the triangles ABQ, APO are similar and also that the triangles ABP, AQO are similar.

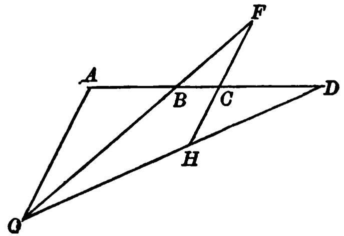
3. If any straight line PQR be drawn touching one given circle at Q and cutting another at P, R, the segments PQ, QR subtend equal or supplementary angles at either of the limiting points of the coaxial system to which the given circles belong.

4. If any straight line $PQRS$ be drawn cutting two given circles of a coaxial system in P, S and Q, R, the segments PQ, RS subtend equal or supplementary angles at either of the limiting points.

ADDITIONAL PROPOSITION 8.

If a pencil be drawn from a point to the four points of a harmonic range and if a straight line be drawn through one of the points parallel to the ray which passes through the conjugate point, the part of the line intercepted between the rays through the other pair of points is bisected at the point.

Let *ABCD* be a harmonic range, and *O* be any point not in the straight line *AD*. Let *OA*, *OB*, *OD* be drawn*, and let *FCH* be drawn parallel to *AO*, meeting *OB*, *OD*, produced if necessary, in *F*, *H*.



Because the triangles *OAB*, *FCB* are equiangular to one another,
OA is to *FC* as *AB* to *BC*; (Prop. 4.)
and because the triangles *OAD*, *HCD* are equiangular to one another,
OA is to *HC* as *AD* to *DC*.
And because *ABCD* is a harmonic range,
AB is to *BC* as *AD* to *DC*;
therefore *OA* is to *FC* as *OA* to *HC*; (V. Prop. 5.)
therefore *FC* is equal to *HC*. (V. Prop. 3.)

NOTE. The converse of this proposition is true, viz. if the line *FH* be bisected at *C*, then *ABCD* is a harmonic range.

EXERCISE.

1. Give a construction to find the fourth point of a harmonic range when three points are given.

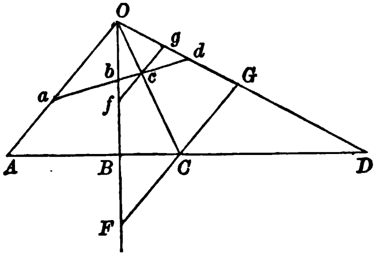
* The ray *OC* of the pencil *O* (*ABCD*) is omitted in the figure, as it is not wanted in the proof. Similar omissions will be met with elsewhere.

ADDITIONAL PROPOSITION 9.

The points, in which a harmonic pencil is cut by any straight line, form a harmonic range.

Let $O\,(ABCD)$ be a pencil drawn through the points of the harmonic range $ABCD$: let $abcd$ be any other straight line cutting the rays OA, OB, OC, OD in a, b, c, d respectively.

Through C, c draw GCF, gcf parallel to OA cutting the rays OB, OD, produced if necessary, in F, G and f, g.



Because $ABCD$ is a harmonic range, and GCF is parallel to OA,
therefore FC is equal to CG. (Add. Prop. 8.)
And because fcg is parallel to FCG,
therefore fc is equal to cg. (Ex. 1, page 369.)
And because fcg is parallel to Oa and fc is equal to cg,
$abcd$ is a harmonic range. (Note, p. 432.)

EXERCISES.

1. The pencil formed by joining the four angular points of a square to any point on the circumscribing circle of the square is a harmonic pencil.

2. Give a construction for drawing the fourth ray of a harmonic pencil, when three rays are given.

3. Draw a harmonic pencil of which the rays pass through the angular points of a rectangle, and one of which is given in direction.

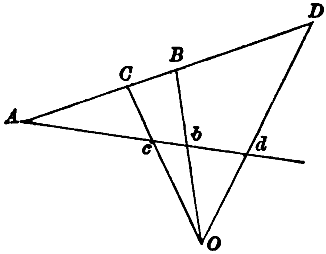
4. CA, CB are two tangents to a circle; E is the foot of the perpendicular from B on AD the diameter through A; prove that CD bisects BE.

ADDITIONAL PROPOSITION 10.

If two harmonic ranges have two corresponding points, one in each range, coincident, the straight lines joining the other pairs of corresponding points pass through a point.

Let $ACBD$, $Acbd$ be two harmonic ranges, of which the point A is a common point.

Draw Cc, Bb and let them, produced if necessary, meet in O, and draw OD.



Because $O\ (ACBD)$ is a harmonic pencil,
if Acb cut OD in d',
then $Acbd'$ is a harmonic range; (Add. Prop. 9.)
therefore Ac is to cb as Ad' to $d'b$;
but $Acbd$ is a harmonic range;
therefore Ac is to cb as Ad to db;
therefore Ad' is to $d'b$ as Ad to db, (V. Prop. 5.)
and d' coincides with d; (Ex. 1, page 359.)
i.e. Cc, Bb, Dd meet in a point.

EXERCISE.

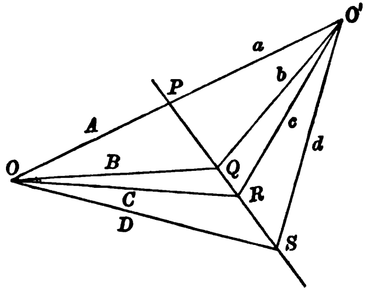
1. Prove that the intersections of the pairs of straight lines Cb, Bc; Bd, Db; Dc, Cd in the above figure lie on a straight line which passes through A.

ADDITIONAL PROPOSITION 11.

If two harmonic pencils have two corresponding rays, one of each pencil, coincident, the intersections of the other three pairs of corresponding rays lie on a straight line.

Let $O(ABCD)$, $O'(abcd)$ be two harmonic pencils, of which $OAaO'$ is a common ray.

Let OB, $O'b$ meet in Q, and OC, $O'c$ in R; draw QR and let it meet OO' in P, OD in S, and $O'd$ in s.



Then because $O(ABCD)$ is a harmonic pencil,

$PQRS$ is a harmonic range; (Add. Prop. 9.)

therefore PQ is to QR as PS to SR;

and because $O'(abcd)$ is a harmonic pencil,

$PQRs$ is a harmonic range;

therefore PQ is to QR as Ps to sR;

therefore Ps is to sR as PS to SR, (V. Prop. 5.)

and the points S, s coincide; (Ex. 1, page 359.)

i.e. the intersections of OB, $O'b$; OC, $O'c$, and OD, $O'd$ are collinear.

EXERCISE.

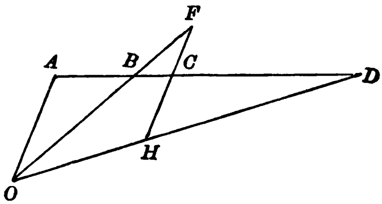
1. Prove that the straight lines joining the intersections of the pairs of straight lines OB, $O'c$ and OC, $O'b$; OC, $O'd$ and OD, $O'c$; OD, $O'b$ and OB, $O'd$ intersect in a point which lies on the line OO'.

ADDITIONAL PROPOSITION 12.

If a pencil be drawn from a point to the four points of an anharmonic range and if a straight line be drawn through one of the points parallel to the ray which passes through a second point, the part of it intercepted between the rays through the other pair of points will be divided in a constant ratio at the first point.

Let $ABCD$ be an anharmonic range and O be any point not in the straight line AD.

Let OA, OB, OD be drawn and FCH be drawn parallel to AO meeting OB, OD, produced if necessary, in F, H.



Because the triangles OAB, FCB are equiangular to one another,

OA is to FC as AB to BC; (Prop. 4.)

and because the triangles OAD, HCD are equiangular to one another,

OA is to HC as AD to DC;

therefore the ratio of the ratio OA to FC to the ratio OA to HC is equal to the ratio of the ratio AB to BC to the ratio AD to DC, which is constant;

i.e. the ratio HC to FC is constant and is equal to the ratio of the range $ABCD$. (Def. 10.)

EXERCISES.

1. If $ABCD$, $ABCE$ be two like anharmonic ranges, then the points D, E coincide.

2. If the ratio of the range $ABCD$ be equal to the ratio of the range $ADCB$, the range $ABCD$ is harmonic.

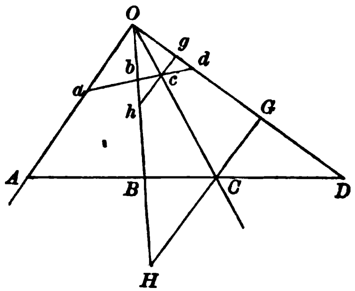
ADDITIONAL PROPOSITION 13.

The points in which an anharmonic pencil is cut by any straight line form an anharmonic range of constant ratio.

Let $O(ABCD)$ be a pencil drawn through the points of the anharmonic range $ABCD$.

Let $abcd$ be any other straight line cutting the rays OA, OB, OC, OD in a, b, c, d respectively.

Through C, c draw GCH, gch parallel to OA cutting the rays OD, OB in G, H and g, h.



Because $ABCD$ is an anharmonic range,
and GCH is parallel to OA,
therefore the ratio of GC to CH is the ratio of the range $ABCD$;
(Add. Prop. 12.)
and because $abcd$ is an anharmonic range and gch is parallel to Oa,
therefore the ratio of gc to ch is the ratio of the range $abcd$;
(Add. Prop. 12.)
and because gch is parallel to GCH,
gc is to ch as GC to CH; (Ex. 2, page 365.)
therefore $abcd$ is an anharmonic range of ratio equal to that of the range $ABCD$.

EXERCISES.

1. Find a point on a given straight line such that lines drawn from it to three given points shall intercept on any parallel to the given line lengths having a given ratio.

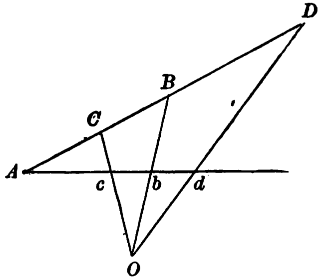
2. Three points F, G, H are taken on the side BC of a triangle ABC: through G any line is drawn cutting AB and AC in L and M respectively; FL and HM intersect in K; prove that K lies on a fixed straight line passing through A.

ADDITIONAL PROPOSITION 14.

If two like anharmonic ranges have two corresponding points, one in each range, coincident, the straight lines joining the other pairs of corresponding points pass through a point.

Let $ACBD$, $Acbd$, be two like anharmonic ranges, of which the point A is a common point.

Draw Cc, Bb, and let them, produced if necessary, meet in O; and draw OD.



Because $O(ACBD)$ is an anharmonic pencil,

if Acb cut OD in d',

then $Acbd'$ is an anharmonic range of ratio equal to that of the pencil; (Add. Prop. 13.)

and $Acbd$ is a like anharmonic range;

therefore the ratio of the ratio Ac to cb to the ratio Ad' to $d'b$ is equal to the ratio of the ratio Ac to cb to the ratio Ad to db;

therefore the ratio Ad' to $d'b$ is equal to the ratio Ad to db,

and d' coincides with d, (Ex. 1, page 359.)

i.e. Cc, Bb, Dd meet in a point.

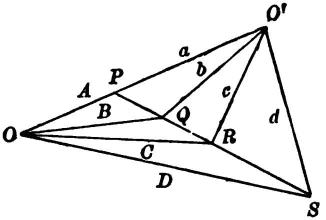
EXERCISE.

1. Prove that the intersections of the pairs of straight lines Cb, Bc; Bd, Db; Dc, Cd in the above figure lie on a straight line through A.

ADDITIONAL PROPOSITION 15.

If two like anharmonic pencils have two corresponding rays, one in each pencil, coincident, the intersections of the other three pairs of corresponding rays lie on a straight line.

Let $O(ABCD)$, $O'(abcd)$ be two like anharmonic pencils, of which $OAaO'$ is a common ray. Let OB, $O'b$ meet in Q, and OC, $O'c$ in R; draw QR and let it meet OO' in P, OD in S and $O'd$ in s.



Because $PQRS$ is a transversal of the anharmonic pencil $O(ABCD)$,
$PQRS$ is a range of ratio equal to that of the pencil; (Add. Prop. 13)
and because $PQRs$ is a transversal of the anharmonic pencil $O'(abcd)$,
$PQRs$ is a range of ratio equal to that of the pencil;
and because $O(ABCD)$, $O'(abcd)$ are like anharmonic pencils,
(Hypothesis)
therefore $PQRS$, $PQRs$ are two like anharmonic ranges;
therefore the points S, s coincide; (Ex. 1, page 436)
i.e. the intersections of OB, $O'b$; OC, $O'c$; and OD, $O'd$ are collinear.

EXERCISE.

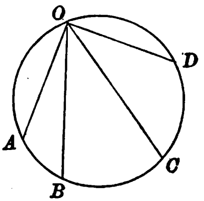
1. Prove that the straight lines joining the intersections of the pairs of straight lines OB, $O'c$ and OC, $O'b$; OC, $O'd$ and OD, $O'c$; OD, $O'b$ and OB, $O'd$ intersect in a point which lies on the straight line OO'.

ADDITIONAL PROPOSITION 16.

The anharmonic ratio of the pencil formed by joining four given points on a circle to any fifth point on the same circle is constant.

Let A, B, C, D be four given points on a circle, and let O be any fifth point on the circle, and let the pencil $O\ (ABCD)$ be drawn.

Take O' any other point in the same arc AD as O;
then the angles $AO'B$, $BO'C$, $CO'D$ are equal to the angles AOB, BOC, COD respectively;
and the pencil $O'\ (ABCD)$ is equal * to the pencil $O\ (ABCD)$.



Next take O' any point in the arc AB.
Then the angles $BO'C$, $CO'D$ are equal to the angles BOC, COD respectively,
and the angle between $O'B$ and AO' produced, say $O'A_1$, is equal to the angle AOB,
and the pencil $O'\ (A_1BCD)$ is equal to the pencil $O\ (ABCD)$.

Similarly it can be proved that the pencil is the same for all positions of O' on the circle.

EXERCISE.

1. The locus of the vertex of a harmonic pencil, whose rays pass through the angular points of a square, is the circumscribed circle of the square.

* In the sense that one pencil can be shifted so that its rays coincide with the rays of the other pencil. (I. Def. 21.)

ADDITIONAL PROPOSITION 17.

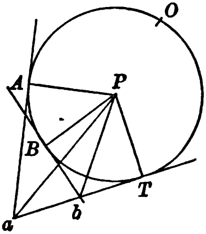
The anharmonic ratio of the range formed by the intersections of four given tangents to a circle by any fifth tangent to the same circle is constant.

Let Aa, Bb, Cc, Dd be the tangents at four given points, A, B, C, D on a circle, and let them cut the tangent at any fifth point T on the circle in a, b, c, d.

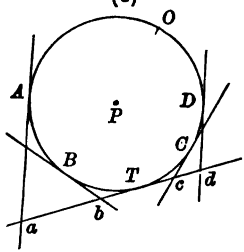
Find P the centre, and take any point O on the circle.

Draw PA, PB, PT, Pa, Pb.

(1)



(2)



Because the angle APT is equal to twice the angle AOT,

(III. Prop. 20)

and also equal to twice the angle aPT,

the angle AOT is equal to the angle aPT.

Similarly it can be proved that

the angle BOT is equal to the angle bPT;

therefore the angle AOB is equal to the angle aPb.

Similarly it can be proved that the angles BOC, COD are equal to the angles bPc, cPd;

therefore the pencil $O\,(ABCD)$ is equal to the pencil $P\,(abcd)$,

and the pencil $O\,(ABCD)$ has a constant ratio; (Add. Prop. 16)

therefore the pencil $P\,(abcd)$ has a constant ratio;

and therefore the range $abcd$ has a constant ratio. (Add. Prop. 13)

EXERCISE.

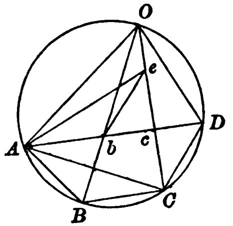
1. If a straight line cut the four sides of a square in a harmonic range, it touches the inscribed circle of the square.

ADDITIONAL PROPOSITION 18.

The anharmonic ratio of the pencil formed by joining four points on a circle to any fifth point on the circle is the same as the ratio of the rectangles contained by the chords that join the points.

Let A, B, C, D be four given points on a circle. Draw AB, AC, AD, BC, CD.

In AD take Ab equal to AB; draw Bb and let it be produced to meet the circle in O; draw OA, OC, OD and let OC cut AD in c; draw be parallel to CD to meet OC in e, and draw Ae.



Because be is parallel to CD,
the angle bec is equal to the angle OCD,
which is equal to the angle OAc; (III. Prop. 21)
therefore A, b, e, O lie on a circle.
Therefore the angle Aeb is equal to the angle AOB,
which is equal to the angle ACB;
and the angle Abe is equal to the supplement of the angle AOe,
which supplement is equal to the angle ABC;
therefore the triangles Abe, ABC are equiangular to one another;
and Ab is equal to AB; (Constr.)
therefore be is equal to BC.

Now the anharmonic ratio of the pencil $O(ABCD)$ is equal to that of the range $AbcD$, (Add. Prop. 13)
which is equal to the ratio of the ratio Ab to bc to the ratio AD to Dc, that is, to the ratio compounded of the ratios Ab to bc and Dc to AD, which is the ratio of the rectangle Ab, cD to the rectangle AD, bc.
(See page 399.)

And because be is parallel to CD,

the triangles DcC, bce are equiangular to one another;

therefore cD is to bc as CD to be; (Prop. 4)

therefore the ratio of the rectangle Ab, cD to the rectangle AD, bc is equal to the ratio of the rectangle AB, CD to the rectangle AD, be.

Therefore the anharmonic ratio of the pencil $O\ (ABCD)$ is equal to the ratio of the rectangle AB, CD to the rectangle AD, BC.

The anharmonic ratio of a range $ABCD$ is defined to be (Definition 10) the ratio of the ratio AB to BC to the ratio AD to DC, which, by Definition 8 of Book V., is equal to the ratio compounded of the ratios AB to BC and DC to AD; and this last ratio has, on page 399, been shewn to be equal to the ratio of the rectangle AB, CD to the rectangle AD, BC.

Now it can be proved (Ex. 1, page 137) that, if $ABCD$ be a range, the sum of the rectangles AB, CD and AD, BC is equal to the rectangle AC, BD.

If therefore any two of these three rectangles be given, the third is at once found. There are six ratios, of which one of these rectangles is the antecedent and another the consequent; if any one of these ratios be given, the other five ratios are at once found. Now in the definition the ratio of the rectangle AB, CD to the rectangle AD, BC is defined as *the* anharmonic ratio of the range $ABCD$. There is no reason against adopting any other of the six ratios as *the* ratio of the range, but it is important strictly to adhere throughout an investigation to one and the same ratio.

EXERCISES.

1. The anharmonic ratio of the range formed by the intersections of four given tangents to a circle with any fifth tangent is equal to the ratio of the rectangles contained by the chords which join the points of contact of the given tangents.

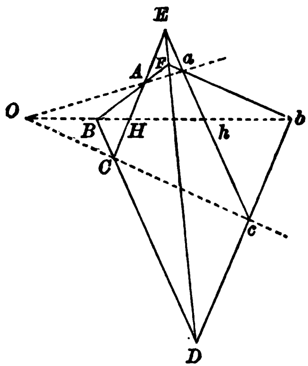
2. Two fixed points D, E are taken on the diameter AB of a circle, and P any point on the circumference; perpendiculars AM, BN are let fall on PD, PE; prove that the ratio of the rectangle PM, PN to the rectangle AM, BN is constant.

ADDITIONAL PROPOSITION 19.

If two triangles be such that the straight lines joining their vertices in pairs pass through a point, the intersections of pairs of corresponding sides lie on a straight line.

Let ABC, abc be two given triangles such that the straight lines Aa, Bb, Cc meet in a point O.

Let the pairs of sides BC, bc; CA, ca; AB, ab, produced if necessary, meet in D, E, F respectively, and let OBb cut AC, ac in H, h.



Because $EAHC$, $Eahc$ cut the same pencil $O(EAHC)$,

$EAHC$, $Eahc$ are like anharmonic ranges; (Add. Prop. 13),

therefore the pencils $B(EAHC)$, $b(Eahc)$ are like anharmonic pencils;

and they have a common ray $BHhb$;

therefore the intersections of the pairs of rays BC, bc; BE, bE; BA, ba lie on a straight line; (Add. Prop. 15);

that is, D, E, F lie on a straight line.

NOTE. The point O is often called **the pole** of the triangles ABC, abc, and the straight line DEF **the axis** of the triangles.

This theorem may then be enunciated thus, *compolar triangles are coaxial.*

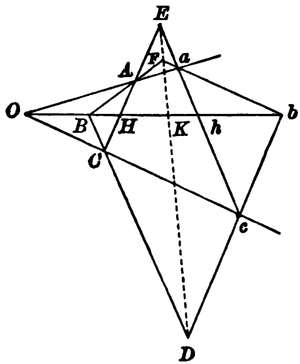
Compolar triangles are often said to be **in perspective**.

ADDITIONAL PROPOSITION 20.

*If two triangles be such that the intersections of their sides taken in pairs lie on a straight line, the straight lines joining pairs of corresponding vertices meet in a point.**

Let *ABC*, *abc* be two given triangles such that the pairs of sides *BC*, *bc*; *CA*, *ca*; *AB*, *ab* intersect in three points *D*, *E*, *F* lying on a straight line.

Let the straight lines *Aa*, *Bb*, *Cc* be drawn and let *Bb* cut *AC*, *ac*, *DEF* in *H*, *h*, *K* respectively.



Because the pencils *B* (*EFKD*), *b* (*EFKD*) have a common transversal *EFKD*,

they are like anharmonic pencils;

and because *EAHC* is a transversal of the pencil *B* (*EFKD*),

and *Eahc* is a transversal of the pencil *b* (*EFKD*),

EAHC, *Eahc* are like anharmonic ranges; (Add. Prop. 13)

and they have a common point *E*;

therefore the lines *Aa*, *Hh*, *Cc* meet in a point, (Add. Prop. 14)

that is, *Aa*, *Bb*, *Cc* meet in a point.

Note. The above theorem may be enunciated thus, *coaxial triangles are compolar.*

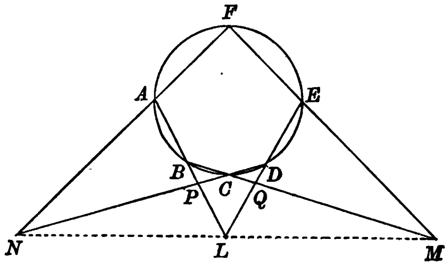
* The theorems of this and the preceding pages are attributed to Gerard Desargues (born at Lyons 1593, died 1662).

ADDITIONAL PROPOSITION 21.

*If a hexagon be inscribed in a circle, the intersections of pairs of opposite sides lie on a straight line.**

Let *ABCDEF* be a hexagon inscribed in a given circle.

Let the pairs of sides *AB*, *DE*; *BC*, *EF*; *CD*, *FA* meet in *L*, *M*, *N* respectively, and let *AB*, *CD* meet in *P*, and *BC*, *DE* in *Q*.



Because *A*, *B*, *C*, *D*, *E*, *F* are points on a circle,
A (*BCDF*), *E* (*BCDF*) are like anharmonic pencils; (Add. Prop. 16)
and because *PCDN* is a transversal of the pencil *A* (*BCDF*),
and *BCQM* is a transversal of the pencil *E* (*BCDF*),
PCDN, *BCQM* are like anharmonic ranges; (Add. Prop. 13)
and they have a common point *C*;
therefore the lines *PB*, *DQ*, *NM* meet in a point, (Add. Prop. 14)
that is, *AB*, *DE*, *NM* meet in a point,
or, in other words, the points *L*, *M*, *N* lie on a straight line.

EXERCISES.

1. If the tangents to the circumscribed circle of a triangle *ABC* at *A*, *B*, *C* meet the sides *BC*, *CA*, *AD*, in *L*, *M*, *N*, then *L*, *M*, *N* lie on a straight line.

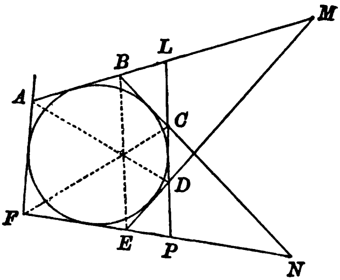
2. If *ACE*, *BDF* be two ranges, then the intersections of the pairs of straight lines *AB*, *DE*; *BC*, *EF*; *CD*, *FA* are collinear.

* This theorem is true for any conic. It was discovered at the age of sixteen by Blaise Pascal (born at Clermont 1623, died at Paris, 1662).

ADDITIONAL PROPOSITION 22.

*If a hexagon be described about a circle, the straight lines joining opposite vertices pass through a point.**

Let $ABCDEF$ be a hexagon described about a given circle.
Let AB meet CD, DE in L, M and EF meet BC, CD in N, P.



Because the six sides of the hexagon are tangents to a circle, the points where the sides AB, EF are cut by the remaining sides of the hexagon form like anharmonic ranges, (Add. Prop. 17)

that is, $ABLM$, $FNPE$ are like anharmonic ranges;

therefore $D(ABLM)$, $C(FNPE)$ are like anharmonic pencils;

and they have a common ray $LCDP$;

therefore the pairs of rays DA, CF; DB, CN; DM, CE intersect on a straight line, (Add. Prop. 15)

that is, DA, CF meet in a point on the line BE,

or, in other words, AD, BE, CF meet in a point.

EXERCISE.

1. If the inscribed circle of a triangle ABC touch the sides BC, CA, AB in D, E, F, then AD, BE, CF meet in a point.

* This theorem also is true for any conic. It was discovered by Charles Julien Brianchon (born at Sèvres, 1785).

SIMILAR FIGURES.

If two similar figures be placed so that their corresponding sides are parallel in pairs, the figures are then said to be **similar and similarly situate.**

It appears from Proposition 32 that, if two triangles be similar and similarly situate, then the lines joining pairs of corresponding vertices meet in a point. It is easily proved that, if two polygons of any number of sides be similar and similarly situate, then the lines joining pairs of corresponding vertices meet in a point, and further that the ratio of the distances of this point from any pair of corresponding points of the two figures is constant, and is equal to the ratio of a pair of corresponding sides of the two figures. Such a point is called in consequence **a centre of similitude** of the two figures.

It is also easily seen that, if a point be a centre of similitude of two figures, it is also a centre of similitude of any two figures similarly described with reference to the first pair of similar figures; for instance, if a point be a centre of similitude of two triangles, it is also a centre of similitude of the circumscribed circles of the triangles, and also a centre of similitude of the inscribed circles and so on.

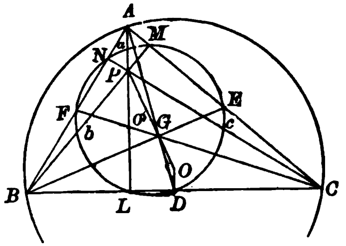
If a pair of corresponding points lie on the same side of the centre of similitude, it is called **a centre of direct similitude**: if a pair of corresponding points lie on opposite sides of the centre of similitude, it is called **a centre of inverse similitude.**

It must be noticed that the centre of direct similitude is at an infinite distance in the case when the two similar figures are equal.

As an example we will prove an important theorem with reference to the centres of similitude of the circumscribed circle and the Nine Point circle of a triangle:

The orthocentre and the centroid of a triangle are the centres of direct and inverse similitude of the circumscribed circle and the Nine Point circle of the triangle.

If *ABC* be a triangle, and *D*, *E*, *F* be the middle points of its sides,
the triangles *ABC*, *DEF* are equiangular to one another,
and the ratio of *AB* to *DE* is equal to the ratio 2 to 1.
(Add Prop., page 101.)
Also the radii of the circles *ABC*, *DEF* are in the ratio of 2 to 1.
Because the lines *AD*, *BE*, *CF* meet in a point *G*,
and the ratios *AG* to *GD*, *BG* to *GE*, *CG* to *GF* are each equal to 2 to 1,
(Add Prop., page 103.)



G is the centre of inverse similitude of the two circles *ABC*, *DEF*.
Again, if *P* be the orthocentre of the triangle *ABC*,
and *a*, *b*, *c* be the middle points of *PA*, *PB*, *PC*,
the ratios *PA* to *Pa*, *PB* to *Pb*, *PC* to *Pc* are each equal to 2 to 1.

Therefore *P* is the centre of direct similitude of the triangles *ABC*, *abc*, and therefore of the circles *ABC*, *abc*.

Now the circles *abc*, *DEF* are identical. (Add. Prop., page 271.)

Therefore *P* is the centre of direct similitude of the circles *ABC*, *DEF*.

Now, if *O*, *O′* be the centres of the circles *ABC*, *DEF*,
since the radii of the circles are in the ratio 2 to 1,
their centres of similitude lie in the straight line *OO′*,
and divide the distance internally and externally in the ratio 2 to 1.

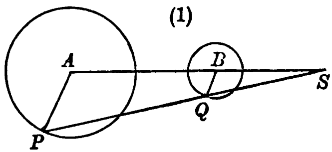
That is, *P*, *O′*, *G*, *O* lie on a straight line,
and *PO* is equal to twice *PO′*,
and *GO* is equal to twice *GO′*.

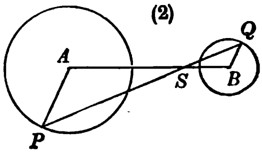
ADDITIONAL PROPOSITION 23.

Every straight line which passes through the extremities of two parallel radii of two fixed circles passes through a centre of similitude of the circles.

Let AP, BQ be two parallel radii of two given circles, whose centres are A, B.

Draw AB, PQ and let them, produced if necessary, meet at S.





Because the triangles APS, BQS are equiangular to one another,

AS is to BS as AP to BQ; (Prop. 4)

that is, S is a point, which divides the distance AB externally (fig. 1) or internally (fig. 2) in the ratio of the radii;

therefore S is a fixed point. (Ex. 1, page 359.)

Again, because the triangles APS, BQS are equiangular to one another, SP is to SQ as AP to BQ; (Prop. 4)

that is, the ratio of the distances SP to SQ is a constant ratio; therefore S is a centre of similitude of the circles.

In figure 1, where the two radii AP, BQ are drawn in the same sense, S is in the line of centres AB produced, and the two distances SP, SQ are drawn in the same direction. S is **a centre of direct similitude.**

In figure 2, where AP, BQ are drawn in opposite senses, S is in the line of centres AB, and the two distances SP, SQ are drawn in opposite directions. S is **a centre of inverse similitude.**

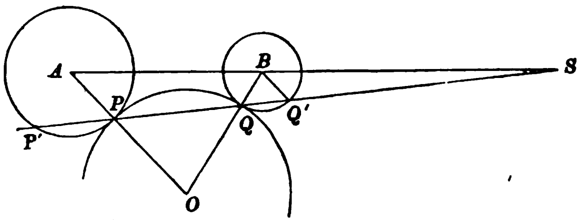
ADDITIONAL PROPOSITION 24.

If a circle be drawn to touch two given circles, the straight line which passes through the points of contact passes through one of the centres of similitude of the given circles.

Let a circle be drawn to touch two given circles, whose centres are A, B, at P, Q.

Draw AP, BQ and let them, produced if necessary, meet at O *.

Draw PQ and let it, produced if necessary, meet the circles again at P', Q': draw BQ'.



Because the circles touch at P, the centre of the circle which is described to touch the given circles must lie in AP;

(III. Prop. 10, Coroll.)

similarly it must lie in BQ;

therefore O is the centre;

therefore the angle OPQ is equal to the angle OQP,

which is equal to the angle BQQ' and therefore to the angle $BQ'Q$;

therefore APO, BQ' are parallel; (I. Prop. 28)

therefore PQ passes through a centre of similitude. (Add. Prop. 23.)

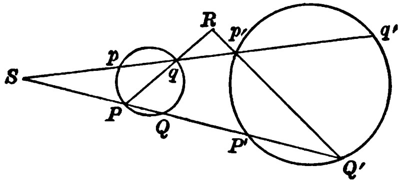
* If AP, BQ be parallel, so that the point O is infinitely distant, and the radius of the circle which touches the given circles at P, Q is infinitely large, PQ becomes one of the common tangents of the given circles, which common tangents pass through S.

ADDITIONAL PROPOSITION 25.

If two straight lines be drawn through a centre of similitude of two given circles to cut the circles, a pair of chords of the two circles joining pairs of inverse points intersect on the radical axis of the given circles, and a pair of chords of the two circles joining pairs of corresponding points are parallel.*

Let S be a centre of similitude of two given circles $PQqp$, $P'Q'q'p'$ and let $SPQP'Q'$, $Spqp'q'$ be any two straight lines drawn through S cutting the circles; P, Q'; Q, P'; p, q'; q, p' being pairs of inverse points and P, P'; Q, Q'; p, p'; q, q' being pairs of corresponding points.

Draw Pq, $Q'p'$, and let them, produced if necessary, meet at R.



Because S is a centre of similitude, Sp is to Sp' as SQ to SQ';

therefore Qp, $Q'p'$ are parallel; (Prop. 2, Part 2)

therefore the angles PQp, $PQ'p'$ are equal; (I. Prop. 29)

and the angles PQp, Pqp are equal; (III. Prop. 21)

therefore the angles $PQ'p'$, Pqp are equal.

Therefore the four points P, q, p', Q' lie on a circle, (III. Prop. 22, Coroll.)

and the rectangle RP, Rq is equal to the rectangle RQ', Rp'; (III. Prop. 36)

therefore the squares on the tangents drawn from R to the two circles are equal,

and consequently R lies on the radical axis of the given circles. (Add. Prop. page 264.)

* For this name see page 460.

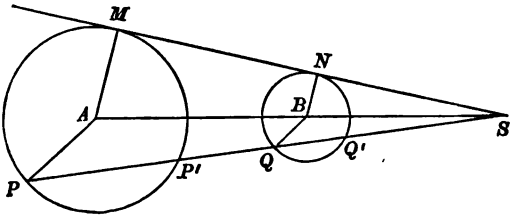
ADDITIONAL PROPOSITION 26.

If a straight line be drawn through a centre of similitude of two given circles, the rectangle contained by the distances of two inverse points is constant.

Let S be a centre of similitude of two given circles, whose centres are A, B.

Draw MNS a common tangent through S (see note on page 451), and let $SQ'QP'P$ be any straight line through S cutting the circles in P, P' and Q, Q'.

Draw AP, BQ, AM, BN.



Because the rectangle SP, SP' is equal to the square on SM,

SP is to SM as SM to SP'. (Prop. 17, Part 2.)

And because S is a centre of similitude,

SP is to SQ as SM to SN;

and therefore SP is to SM as SQ to SN; (V. Prop. 9)

therefore SQ is to SN as SM to SP'; (V. Prop. 5)

therefore the rectangle SQ, SP' is equal to the rectangle SM, SN. (Prop. 16, Part 1.)

Similarly it can be proved that the rectangle SP, SQ' is equal to the rectangle SM, SN.

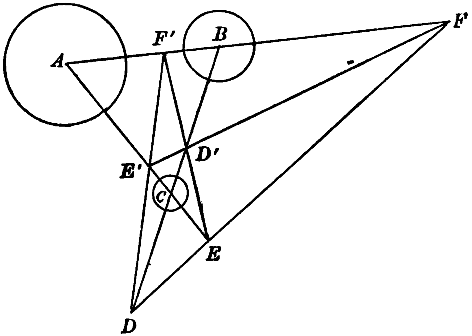
EXERCISE.

1. Prove that in the above figure the rectangle PQ, $P'Q'$ is equal to the square on MN.

ADDITIONAL PROPOSITION 27.

The six centres of similitude of three given circles taken in pairs lie three by three on four straight lines.*

Let A, B, C be the centres of three given circles; let a, b, c be the radii of the A, B, C circles, and let D, D' be the centres of direct and of inverse similitude of the B, C circles, E, E' those of the C, A circles, and F, F' those of the A, B circles.



Because BD is to DC as b to c,

and CE is to EA as c to a,

and AF is to FB as a to b, (Add. Prop. 23)

therefore the ratio compounded of the ratios

BD to DC, CE to EA and AF to FB,

is equal to the ratio compounded of the ratios

b to c, c to a and a to b, that is to unity;

therefore DEF is a straight line. (Add. Prop. 2.)

Similarly it can be proved that $DE'F'$, $D'EF'$, $D'E'F$ are straight lines,

and also that the lines of each of the sets AD', BE', CF'; AD', BE, CF; AD, BE', CF; AD, BE, CF' meet in a point. (Add. Prop. 4.)

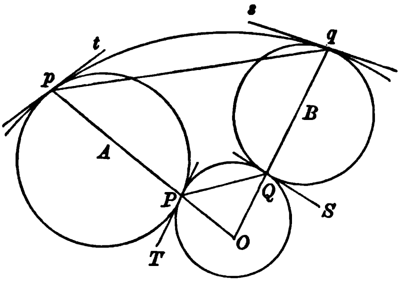
* These lines are called the axes of similitude of the three circles.

ADDITIONAL PROPOSITION 28.

If a point on the radical axis of two given circles be joined to the points of contact of a circle, which touches both the given circles, by straight lines which cut the circles again, another circle can be described to touch the given circles at the points of section.

Let O be a point on the radical axis of the given circles A, B; and let a circle touch them at P, Q.

Draw OP, OQ and let them meet the A, B circles in p, q. Draw PT, QS, pt, qs the tangents to the circles at P, Q, p, q and draw PQ, pq.



Because O is on the radical axis of the circles A, B,
the rectangle OP, Op is equal to the rectangle OQ, Oq;
therefore P, Q, q, p lie on a circle. (Ex. 1, page 253.)
The difference of the angles tpO, sqO is equal to
the difference of the angles TPO, SQO,
which is equal to the difference of the angles PQO, QPO,
which is equal to the difference of the angles qpO, pqO;
(III. Prop. 22)
therefore the angle tpq is equal to the angle sqp.
It is therefore possible to describe a circle touching the circles A, B at p, q.

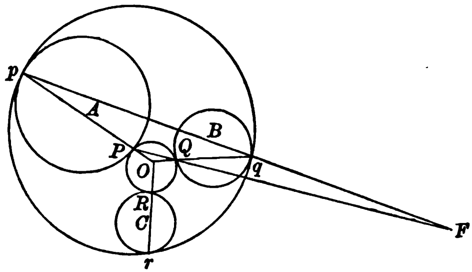
COROLLARY. *If the radical centre of three given circles be joined to the points of contact of a circle, which touches all the three given circles, by straight lines which cut the circles again, another circle can be described to touch the given circles at the points of section.*

(See Ex. 140, page 284.)

ADDITIONAL PROPOSITION 29.

If two circles be drawn to touch three given circles, so that the straight line joining the two points of contact on each of the given circles passes through the radical centre of the given circles, the radical axis of the pair of circles is one of the axes of similitude of the three given circles.

Let PQR, pqr be a pair of circles touching three given circles A, B, C at P, p; Q, q; R, r, so that Pp, Qq, Rr pass through O the radical centre of the circles A, B, C. (Add. Prop. 28, Coroll.)



Draw PQ, pq and let them, produced if necessary, meet in F.

Because the circles PQR, pqr touch the circles A, B,

the lines PQ, pq pass through one of the centres of similitude of the circles A and B; (Add. Prop. 24)

therefore the rectangle FP, FQ is equal to the rectangle Fp, Fq; (Add. Prop. 26.)

therefore F is a point on the radical axis of the circles PQR, pqr.

Similarly it can be proved that a centre of similitude of each of the pairs of circles B, C and C, A lies on the radical axis of PQR, pqr.

Therefore the radical axis of the circles PQR, pqr is an axis of similitude of the circles A, B, C.

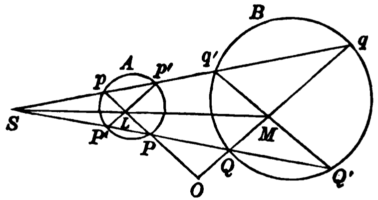
ADDITIONAL PROPOSITION 30.

If a pair of chords of two given circles intersect in the radical axis, and if the chords intersect the polars of one of the centres of similitude in two points collinear with the centre of similitude, then the points of intersection of the chords with the circles lie two by two on two straight lines through the centre of similitude.

Let S be one of the centres of similitude of two given circles A, B; let L, M be two points collinear with S, on the polars of S with respect to the circles A, B; and let O be a point on the radical axis.

Draw OL, and let it intersect the circle A in P, p.

Draw SP, Sp, and let them meet the circle A in P', p', and let the corresponding points to P, P', p, p', where the lines meet the circle B be Q', Q, q', q.



Because L is on the polar of S, the line $P'p'$ passes through L.
(Add. Prop. page 261.)

Because S is the centre of similitude, and the intersections of Pp, $P'p'$, and of $Q'q'$, Qq are corresponding points, and Pp, $P'p'$ intersect at L,

and L, M are corresponding points,

therefore Qq, $Q'q'$ intersect at M;

and because S is the centre of similitude,

Pp, Qq intersect on the radical axis, (Add. Prop. 25)

that is, Qq must pass through O; and it also passes through M.

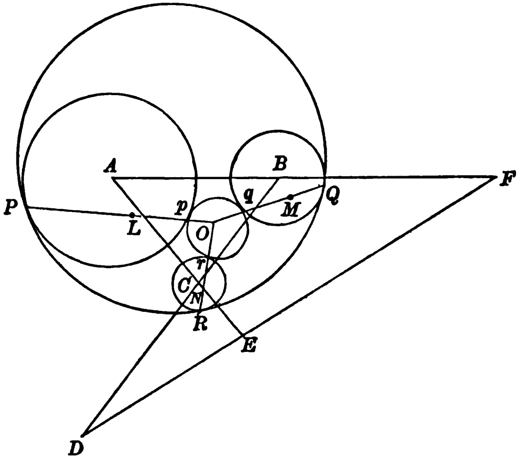
Therefore the points Q, q obtained by the above construction are the points where the straight line OM cuts the circle B, that is, if OL cut the circle A in P, p and OM cut the circle B in Q, q, then PQ, pq pass through S.

ADDITIONAL PROPOSITION 31.

*To describe a circle to touch three given circles.**

Let A, B, C be three given circles. Find O the radical centre of the circles A, B, C, and draw DEF one of their axes of similitude.

Find the poles L, M, N of the line DEF with respect to the circles A, B, C respectively. Draw OL, OM, ON and let them, produced if necessary, cut the circles A, B, C respectively in P, p; Q, q; R, r.



Because L, M are the poles of DEF with respect to the circles A, B, therefore L, M are on the polars of F with respect to the circles A, B;

and because F is a centre of similitude of the two circles,

therefore LMF is a straight line.

Therefore the line PQ passes through F. (Add. Prop. 30.)

* This solution of the problem is due to Joseph Diez Gergonne (born at Nancy 1771, died at Montpellier 1859).

Therefore a circle can be described through P, Q to touch the given circles at P, Q. (Add. Prop. 24.)

Similarly it can be proved that circles can be described through Q, R, and through R, P to touch at each pair of points.

Therefore these three circles are identical (see Ex. 140, page 284),

that is, the circle PQR touches the given circles at P, Q, R.

Similarly the circle pqr touches the given circles at p, q, r.

Since there are four axes of similitude, and since two circles can be obtained by the foregoing construction from each axis of similitude, there are 2×4 or 8 circles which can be described to touch three given circles.

It is readily seen that, if three circles touch a fourth circle, there are two distinct possible types of configuration of the three circles relatively to the circle which they touch :

(1) the three circles may lie on the same side of the fourth circle,

(2) two of the three circles may lie on one side and the third on the other side of the fourth circle.

Since any one of the three given circles may be the one which lies by itself on the one side of the fourth circle, the type (2) may be subdivided into three different groups.

It can be proved without much difficulty that of the eight circles which can be described to touch three given circles A, B, C, there are four pairs of circles, such that

A, B, C lie on the same side of each circle of one pair;

A lies on one side, and B, C on the other, of each circle of a second pair;

B lies on one side, and C, A on the other, of each circle of a third pair;

and C lies on one side, and A, B on the other, of each circle of a fourth pair.

Each of these four pairs of circles is obtained by the above construction from one of the axes of similitude of the given circles.

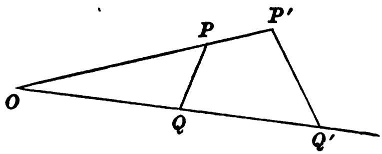
INVERSION.

1. We will now proceed to give an account of a geometrical Method called Inversion, and we will do so without stating the theorems which we are about to establish in the formal way in which theorems have been stated heretofore. We will further depart from our former method by making use of the arithmetical method of representing geometrical magnitudes (see page 135), so that we shall not be debarred from using fractions to represent the ratios of geometrical quantities, and if necessary we shall use the signs ordinarily used in Algebra to signify addition, subtraction, and the other elementary operations.

2. Definition. If O be a fixed point and P any other point, and if on the straight line OP (produced if necessary) we take a point P' such that

$$OP \cdot OP' = a^2,$$

where a is a constant, then each of the points P, P' is called **the inverse** of the other with respect to the circle whose centre is O and radius a.



The straight line OP is often called **the radius vector** of the point P.

The point O is called **the pole of inversion** and a **the radius of inversion.**

If the point P trace out a curve, the curve which is the locus of P' is called **the inverse of the curve** which is the locus of P.

3. If we consider the points P', P'', which are the inverses of P with respect to the same pole and different radii of inversion a, b,

since $$OP \,.\, OP' = a^2 \text{ and } OP \,.\, OP'' = b^2,$$

therefore $$OP' : OP'' = a^2 : b^2.$$

It follows that since the points P', P'' lie on the same radius vector, and $OP' : OP''$ is a constant ratio, the loci of P', P'' are two similar curves of which O is a centre of similitude.

4. If P, P', and Q, Q' be two pairs of inverse points, then since

$$OP \,.\, OP' = a^2 \text{ and } OQ \,.\, OQ' = a^2,$$

therefore $$OP \,.\, OP' = OQ \,.\, OQ';$$

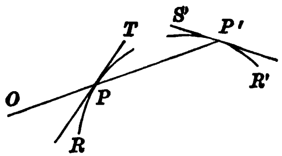
it follows that P, Q, Q', P' lie on a circle, and the angle OPQ is equal to the angle $OQ'P'$.

Again, if the line OQQ' approach nearer and nearer to the position of OPP', the lines QP, $Q'P'$ in the limit are the tangents to the curves which are the loci of P and P' at P and P' respectively.

(See page 217.)

We may state this result in the form :

Two inverse curves at two inverse points cut the radius vector through the pole of inversion at the same angle on opposite sides.



It follows that *any two curves cut at the same angle as their inverse curves at the inverse point.* (See Definition on page 266.)

5. From the definition of inversion it follows at once that *a straight line through the pole inverts into itself.*

Three points A, B, C in the same radius vector, whose distances OA, OB, OC are such that $OA + OC = 2OB$, invert into three points A', B', C', whose distances OA', OB', OC' are such that

$$\frac{1}{OA'} + \frac{1}{OC'} = \frac{2}{OB'}.$$

Three magnitudes such as OA, OB, OC are said to be in **Arithmetical Progression**, and three magnitudes such as OA', OB', OC' are said to be in **Harmonical Progression**.

Because $$\frac{1}{OA'}+\frac{1}{OC'}=\frac{2}{OB'},$$

therefore $OB' . OC'+OA' . OB'=2OA' . OC'$;

whence $OA' . B'C'=OC' . A'B'$,

and $OA' : A'B'=OC' : C'B'$;

therefore $OA'B'C'$ is a harmonic range.

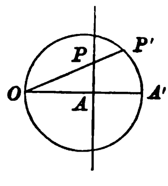
Also three points A, B, C in the same radius vector, whose distances OA, OB, OC are such that $OA . OC=OB^2$, invert into three points A', B', C', whose distances are such that $OA' . OC'=OB'^2$.

Three magnitudes such as OA, OB, OC are said to be in **Geometrical Progression.**

It may be noticed that three magnitudes a, b, c are in Arithmetical, Geometrical, or Harmonical Progression according as

$$a-b : b-c=a : a,$$
$$a-b : b-c=a : b,$$
or $$a-b : b-c=a : c \text{ respectively.}$$

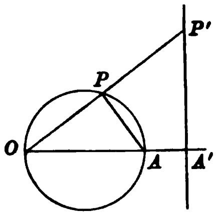
6. Let P be a point on a fixed straight line; draw OA perpendicular to the line, and on OA, produced if necessary, take the point A' such that $OA . OA'=a^2$.



Let P' be the inverse of P, so that $OP . OP'=a^2$: then $OP . OP'=OA . OA'$; therefore P, A, A', P' lie on a circle, and the angle $OP'A'$=the angle OAP=a right angle; therefore the locus of P' is a circle whose diameter is OA'. Hence the theorem :

A straight line which does not pass through the pole of inversion inverts into a circle which passes through the pole and touches there a parallel to the given line.

7. Again let P be any point on a circle, of which OA is a diameter. Take in OA, produced if necessary, the point A' such that $OA \cdot OA' = a^2$.



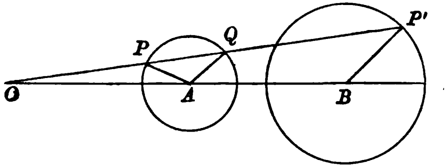
Let P' be the inverse of P, so that $OP \cdot OP' = a^2$;

then $OP \cdot OP' = OA \cdot OA'$;

therefore P, A, A', P', lie on a circle, and the angle $P'A'O$ = the angle OPA = a right angle, i.e. the locus of P' is a straight line through A' the inverse of A drawn at right angles to OA'. Hence the theorem:

A circle which passes through the pole of inversion inverts into a straight line parallel to the tangent at the pole.

8. Again, let P be any point on a circle, which does not pass through the pole, and P' be the inverse point so that $OP \cdot OP' = a^2$: let OP, produced if necessary, cut the circle in Q; find A the centre and draw QA, and let OA, produced if necessary, cut $P'B$ drawn parallel to QA in B.



Because $OP \cdot OQ = t^2$, where t represents the length of the tangent from O to the given circle, and $OP \cdot OP' = a^2$,

therefore $\quad OP' : OQ = a^2 : t^2 = OB : OA = BP' : AQ$;

therefore the locus of P' is a circle such that B is its centre and O is one of the centres of similitude of it and the given circle.

Hence the theorem: *A circle, which does not pass through the pole of inversion, inverts into a circle which does not pass through the pole, and is such that the pole is a centre of similitude of the two circles.*

9. Again, if P, P' and Q, Q' be two pairs of inverse points,

$$OP \cdot OP' = OQ \cdot OQ' = a^2,$$

the triangles OPQ, $OQ'P'$ are similar.

Therefore
$$\frac{P'Q'}{PQ} = \frac{OQ'}{OP} = \frac{OQ' \cdot OQ}{OP \cdot OQ} = \frac{a^2}{OP \cdot OQ},$$

or
$$P'Q' = a^2 \cdot \frac{PQ}{OP \cdot OQ}.$$

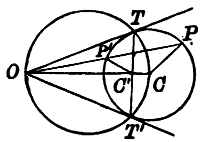
Thus the distance between two points in a figure is expressed in terms of the distance between their inverse points and the distances of the inverse points from the pole.

10. Now let us take the ordinary definition of a circle, i.e. the locus of a point P such that its distance PC from a fixed point C is constant.

Let P' be the inverse point of P; take C' the inverse point of C.

If we please we may in this investigation choose the radius of inversion (see page 461) equal to the tangent drawn from O to the original circle; then P' lies on the same circle as P,

and
$$OC \cdot OC' = OT^2 = OP \cdot OP'.$$



Therefore the triangles $OP'C'$, OCP are similar,

and
$$OP' : P'C' = OC : CP.$$

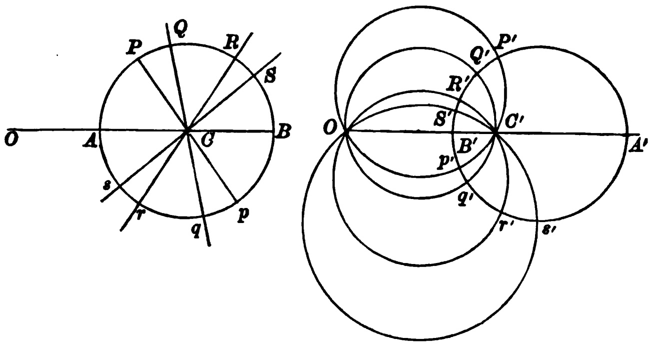
Hence the theorem (which has been already proved otherwise, Add. Prop. 5):

The locus of a point the distances of which from two fixed points are in a constant ratio is a circle.

It will be seen that the straight line TT', which is the polar of O, and the circle on OC as diameter are inverses of each other.

11. Again, it is known from Book III. that all straight lines which cut a circle at right angles pass through the centre, and also that all straight lines through the centre cut the circle at right angles.

If therefore we invert a figure consisting of a series of straight lines cutting a series of concentric circles (centre C) at right angles, we shall obtain a series of coaxial circles passing through O the pole of inversion and through C' the inverse point of C, and cutting each of a second series of circles at right angles. (Add. Prop. page 268.)



Hence the theorem:

A system of concentric circles inverts into a system of coaxial circles.

In the above diagram O is the pole of inversion, and the tangent from O to the circle PQR (or $P'Q'R'$) is taken as the radius of inversion so that the circle inverts into itself. The points O, A, B in the left-hand figure are supposed to coincide with O, B', A' respectively in the right-hand figure: the figures are drawn apart merely for the sake of clearness.

Further, if we invert the last system with respect to any point, we shall get a system of exactly the same nature; viz. a system of circles passing through two fixed points and cutting each of another system of circles at right angles.

Hence the theorem:

A system of coaxial circles inverts into another system of coaxial circles, and the limiting points into the limiting points.

12. A circle can be inverted into itself with respect to any point as pole of inversion, if the tangent to the circle from the point be taken as the radius of inversion.

Any two circles can be inverted into themselves with respect to any point on their radical axis as the pole of inversion.

Any three circles can be inverted into themselves with respect to their radical centre as the pole of inversion.

Hence we conclude that when one circle is drawn to touch two given circles at P and Q, if O be any point on the radical axis of the given circles, the lines OP, OQ will cut the circles again in two points P', Q', such that another circle can be described to touch the given circles at P', Q'.

We also conclude that when one circle is drawn to touch three given circles at P, Q, R, if O be the radical centre, the lines OP, OQ, OR cut the circles again in three points P', Q', R' such that a circle described through them will touch the circles at P', Q', R'.

These two theorems have been proved before (page 455).

13. If two circles PQO, pqO be described to touch two given circles A, B at P, p; Q, q, and to touch each other at O, then the centre of similitude F, which is the point of intersection of the straight lines PQ, pq, must lie on the common tangent to the circles at O.

(Add. Prop. 29.)

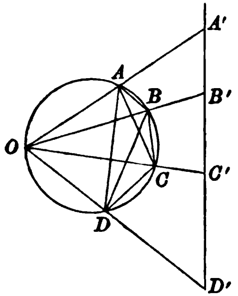
Hence, if O be taken as the pole of inversion, the two circles OPQ, Opq will invert into two parallel common tangents to the inverse circles (page 423), and therefore the given circles will invert into a pair of equal circles.

And since $FO^2 = FP \cdot FQ = FM \cdot FN$, if FMN be a common tangent to the given circles, (Add. Prop. 26) we see that the pole O may be chosen anywhere on the circle, whose centre is F and whose radius is a mean proportional between the tangents from the point F to the circles A, B.

Hence with any point on two definite circles as pole of inversion we can invert two given circles into two equal circles.

It follows that with any one of certain definite points as pole of inversion, *three given circles can be inverted into three equal circles.*

14. If we take A', B', C' three points in order on a straight line, the relation $A'B'+B'C'=A'C'$ exists between the segments.



If we invert with respect to any pole O, the three inverse points A, B, C will lie on a circle through O, and the chords will satisfy the relation

$$\frac{AB}{OA \cdot OB} + \frac{BC}{OB \cdot OC} = \frac{AC}{OA \cdot OC}, \quad \text{(see page 464)},$$

or

$$AB \cdot OC + BC \cdot OA = AC \cdot OB,$$

which is Ptolemy's Theorem. (III. Prop. 37 B.)

Again if we take A', B', C', D', four points in order on a straight line, the relation

$$A'B' \cdot C'D' + A'D' \cdot B'C' = A'C' \cdot B'D'$$

exists between the segments. (Ex. 1, page 137.)

If we invert with respect to any pole O, the four inverse points A, B, C, D will lie on a circle through O, and the chords will satisfy the relation,

$$\frac{AB}{OA \cdot OB} \cdot \frac{CD}{OC \cdot OD} + \frac{AD}{OA \cdot OD} \cdot \frac{BC}{OB \cdot OC} = \frac{AC}{OA \cdot OC} \cdot \frac{BD}{OB \cdot OD},$$

or

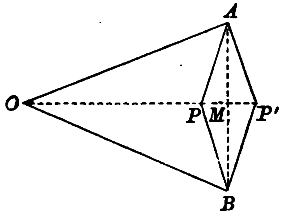
$$AB \cdot CD + AD \cdot BC = AC \cdot BD,$$

which also is a form of Ptolemy's Theorem.

PEAUCELLIER'S CELL.

15. Before we leave the subject of inversion, it will be as well to explain the nature of a simple piece of mechanism, by the use of which the inverse of any given curve can be drawn.

The figure represents such an instrument; OA, OB are two equal rods and AP, PB, BP', $P'A$, four other equal rods, all six being freely hinged together at the points O, A, B, P, P'.



This instrument is generally called a **Peaucellier's Cell***.

Because $APBP'$ is a rhombus,

its diagonals bisect each other at right angles at M; (Ex. 1, page 39)

and because OAB is an isosceles triangle,

and M is the middle point of AB,

OM is at right angles to AB.

Therefore OPP' is a straight line,

and the rectangle $OP \,.\, OP' = OM^2 - MP^2$ (II. Prop. 6)

$= OA^2 - AP^2$

$=$ a constant.

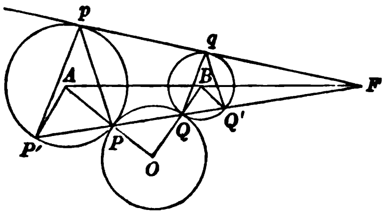
If therefore the point O be fixed, and P be made to trace out any curve, P' will trace out the inverse curve.

* This mechanical invention is due to A. Peaucellier, Capitaine du Génie (à Nice), who proposed the design of such an instrument as a question for solution in the *Nouvelles Annales* 1864 (p. 414). His solution was published in the *Nouvelles Annales* 1873 (pp. 71—8).

We will now proceed to prove a theorem which establishes a relation between six of the common tangents of pairs of four circles which touch a fifth given circle, of a kind similar to that which Ptolemy's Theorem establishes between the chords joining the points of contact.

Let a circle whose centre is O touch two given circles whose centres are A, B at P, Q; let PQ, produced if necessary, cut the circles A, B again in P', Q' and pass through their centre of similitude F; and let pqF be one of their common tangents through F.

(Add. Prop. 24 note.)



Because F is a centre of similitude of the circles,

pP' is parallel to qQ, and pP to qQ'; (Add. Prop. 25)

therefore $pq : P'Q = Fq : FQ$,

and $pq : PQ' = Fq : FQ'$;

therefore $pq^2 : P'Q \,.\, PQ' = Fq^2 : FQ \,.\, FQ'$;

and $Fq^2 = FQ \,.\, FQ'$;

therefore $pq^2 = P'Q \,.\, PQ'$. (Ex. 1, page 453.)

Again because F is a centre of similitude,

OPA is parallel to $Q'B$, and OQB to $P'A$;

therefore $OA : OP = P'Q : PQ$,

and $OB : OQ = PQ' : PQ$;

therefore $OA \,.\, OB : OP \,.\, OQ = PQ' \,.\, P'Q : PQ^2$,

or $OA \,.\, OB : OP^2 = pq^2 : PQ^2$;

therefore, if $OA \,.\, OB = OL^2$,

then $OL : OP = pq : PQ$. (V. Prop. 16.)

Similarly, if two other circles, whose centres are C, D, touch the same circle, centre O, at R, S, and rs be their common tangent,

$$OC \cdot OD : OP^2 = rs^2 : RS^2;$$

therefore, if $OC \cdot OD = OM^2$,

then $OM : OP = rs : RS$; (V. Prop. 16)

therefore $OL \cdot OM : OP^2 = pq \cdot rs : PQ \cdot RS$.

If we denote the common tangent to the two circles which touch the fifth circle at P, Q by (PQ), and so on for the other pairs of circles, we may write this proportion,

$$OL \cdot OM : OP^2 = (PQ) \cdot (RS) : PQ \cdot RS.$$

It can be proved in a similar manner that,

if $OL'^2 = OA \cdot OC$ and $OM'^2 = OB \cdot OD$,

then $OL' \cdot OM' : OP^2 = (PR) \cdot (QS) : PR \cdot QS$.

Now $OL^2 : OL'^2 = OA \cdot OB : OA \cdot OC = OB : OC$,

and $OM'^2 : OM^2 = OB \cdot OD : OC \cdot OD = OB : OC$;

therefore $OL^2 : OL'^2 = OM'^2 : OM^2$,

and $OL : OL' = OM' : OM$; (V. Prop. 16)

therefore $OL \cdot OM = OL' \cdot OM'$. (Prop. 16.)

Hence we have

$$(PQ) \cdot (RS) : PQ \cdot RS = (PR) \cdot (QS) : PR \cdot QS$$

and similarly $= (PS) \cdot (QR) : PS \cdot QR$.

And because, if $PQRS$ be a convex quadrilateral,

$PR \cdot QS = PQ \cdot RS + PS \cdot QR$ (Ptolemy's theorem),

therefore $(PR) \cdot (QS) = (PQ) \cdot (RS) + (PS) \cdot (QR)$.

This theorem is generally known as **Casey's Theorem** *.

It may be observed that the common tangent of each pair of circles, which appears in the equation, is that tangent which passes through the same centre of similitude of the circles as the chord joining the points of contact of the circles with the fifth circle.

* This theorem was discovered by John Casey (born at Kilbenny, County Cork, 1820, died at Dublin 1890).

It is readily seen that, if four circles touch a fifth circle, there are three distinct possible types of configuration of the four circles relatively to the circle which they touch;

(1) the four circles may lie on the same side of the fifth circle,

(2) three of the four circles may lie on one side and the fourth circle on the other side,

(3) two of the four circles may lie on one side and the other two on the other side.

The converse of Casey's Theorem may be stated in the following manner: *if an equation of the form of the equation in Casey's Theorem exist between the common tangents of four circles taken in pairs, the common tangents being chosen in accordance with one of the three possible types of configuration, then the four circles touch a fifth circle.*

The truth of this converse theorem which is often assumed without any attempt at proof can be proved, but the proof of it is thought to be beyond the scope of this work.

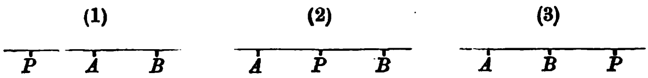
CONTINUITY.

Let us consider a variable point P on a given straight line, on which A, B are two fixed points. It is seen at once that,

(1) if P be outside AB beyond A, then the excess of PB over PA is equal to AB;

(2) if P be in AB, then the sum of AP and PB is equal to AB, and

(3) if P be outside AB beyond B, then the excess of AP over BP is equal to AB.



We can write these results in the forms

(1) $PA+AB=PB$,

(2) $AB=AP+PB$,

(3) $AB+BP=AP$.

Here we observe that, while P changes from one side of A to the other, the distance PA, which vanishes when P coincides with A, changes sides in the equation, which otherwise remains unchanged; and again that while P changes from one side of B to the other, the distance PB similarly changes sides in the equation.

A geometrical theorem consists, in many cases, of a proof that a certain equation exists between a number of geometrical magnitudes, such equation remaining unchanged in form for variations in the geometrical magnitudes involved, consistent with the conditions to which they are subject.

It is found in many of such theorems, as in the illustration which we have just given, that, if subject to continuous variation of some chosen geometrical magnitude some other magnitude continuously diminish and vanish, then in the equation which applies to the configuration determined by the next succeeding values of the chosen variable magnitude, the magnitude which has vanished appears on the opposite side of the equation. This fact is due to the absence of any sudden changes in the magnitudes under consideration. The general law that no sudden change occurs is often spoken of as THE PRINCIPLE OF CONTINUITY.

Let us consider a variable point P on a given straight line, on which A is a fixed point; and let us consider any equation between variable geometrical magnitudes, one of which is PA the distance between P and A. The principle of continuity leads us to expect that, if P in the variation of its position pass from one side to the other of A, the sign of PA in the equation will change. In other words, we may consider the equation to remain unchanged in form, if we resolve to represent by the expression PA not only the distance between P and A, but also the fact that the distance is measured from P towards A. This result is at once obtained by resolving that $-PA$ shall represent a distance equal to PA and measured in the opposite direction; in other words, that $AP = -PA$.

Let us return to the consideration of a variable point P on a given straight line, on which A, B are two fixed points. It appears that the equation which exists between the distances between the points takes different forms according as P is (1) in BA produced, (2) in AB, or (3) in AB produced.

If we allow the use of the *minus* sign, we may write these equations,

$$(1)\quad AB + PA - PB = 0,$$

$$(2)\quad AB - AP - PB = 0,$$

$$(3)\quad AB - AP + BP = 0,$$

where each symbol such as AB represents merely the length of a line measured in the same direction as AB.

It is at once seen that, if we adopt the convention that

$$PQ = -QP,$$

all these equations are the same; each may be written

$$AB = AP + PB,$$

or

$$AB + BP + PA = 0.$$

The first form expresses that the operation of passing from A to B is the same as passing from A to P and then from P to B;

the second form expresses that the aggregate result of the operations of passing from A to B, and then from B to P and then from P to A is to arrive at the point A of starting;

both of which facts are true for *all* combinations of three points A, B, P on a straight line.

The results of the theorems contained in Propositions 5 and 6 of Book II. become the same, if we take into account the fact that the distance BD is measured in opposite directions in the two figures: and similarly the results of Propositions 12 and 13 of Book II. become the same, if we take into account the sign of CD.

As a further illustration of the Principle of Continuity we w[illegible] take Ptolemy's Theorem.

Let us consider a variable point P on a circle, on which A, B, C are three fixed points. It is proved in III. Prop. 37 B, that

(1) if P be in the arc AB,

$$AB.PC=BC.PA+CA.PB;$$

(2) if P be in the arc BC,

$$BC.PA=CA.PB+AB.PC;$$

and

(3) if P be in the arc CA,

$$CA.PB=AB.PC+BC.PA.$$

These equations may be written

$$(1)\quad AB.PC-BC.PA-CA.PB=0,$$
$$(2)\quad AB.PC-BC.PA+CA.PB=0,$$
$$(3)\quad AB.PC+BC.PA-CA.PB=0.$$

Hence, while P passes along the arc from one side of B to the other, the sign of PB, which vanishes when P coincides with B, changes sign in the equation, which otherwise remains unchanged, and so on for passage through C or A.

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